GReCO seminar, IAP
June 2021, *virtual*

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GEOMETRICAL ASPECTS OF STOCHASTIC INFLATION

A PATH (INTEGRAL) TO THE DISCRETISATION AMBIGUITY AND ITS RESOLUTION

[L. Pinol, S. Renaux-Petel, Y. Tada 2018] *Classical and Quantum Gravity 36 no.7, (2019) 07LT01*

CQG Highlights of 2019-2020

[L. Pinol, S. Renaux-Petel, Y. Tada 2020] *Journal of Cosmology and Astroparticle Physics, JCAP04(2021)048*

GEODESI

Thesis defence on June 25th!

European Research Council
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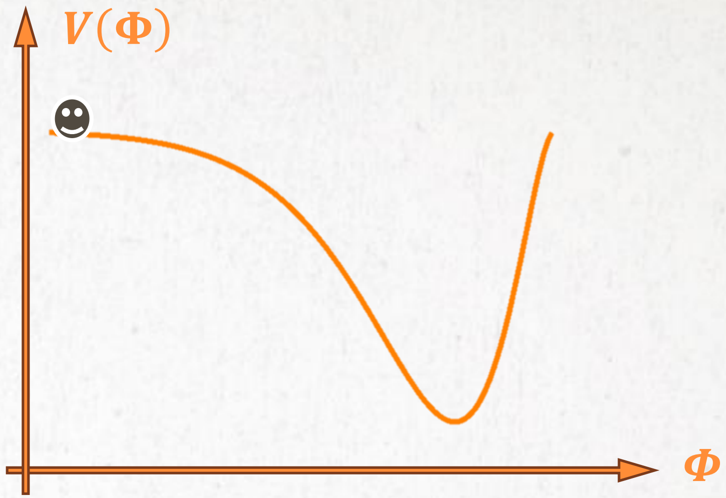
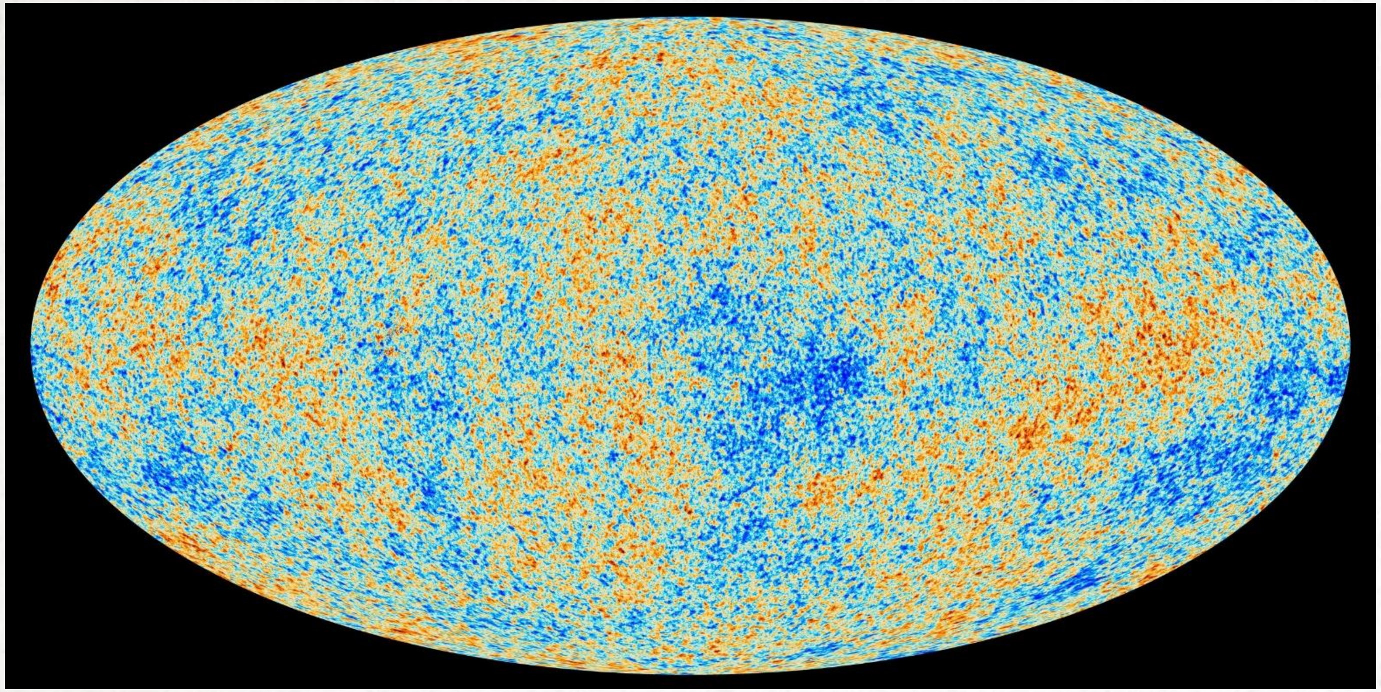
I. STANDARD APPROACH TO INFLATION

A classical background…
… and quantum perturbations



CMB OBSERVATION MOTIVATES INFLATION



$$T \sim 2.73K \; ; \; \frac{\delta T}{T} \sim 10^{-5} \; ; \; |\Omega_k| \ll 1$$

- How is the universe so homogeneous?
  Horizon problem

- Why is the universe so spatially flat?
  Flatness problem

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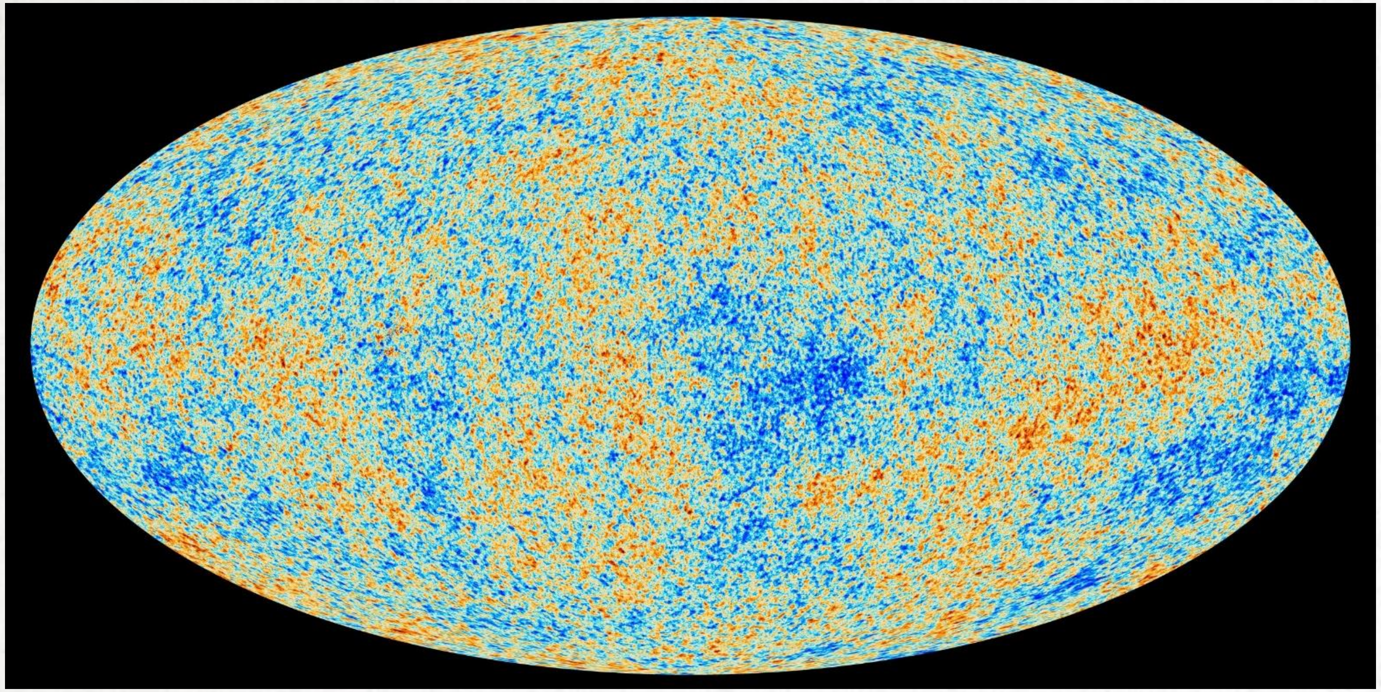

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  Horizon problem

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  Flatness problem

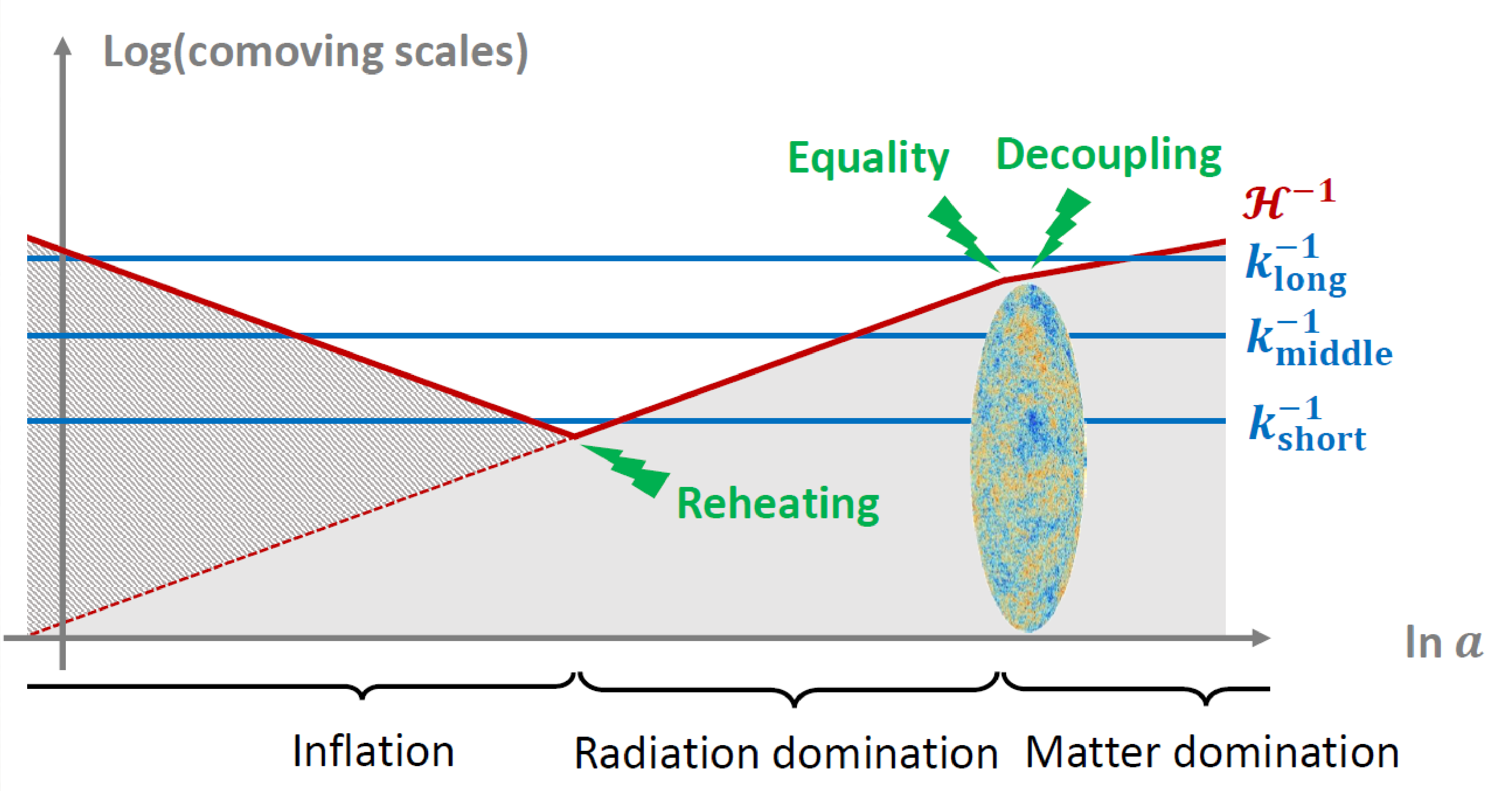
Inflation, an era of accelerated expansion of the Universe, solves both the horizon and flatness problems

$$N_{\mathrm{inf}} = \ln\left(\frac{a_{\mathrm{end}}}{a_{\mathrm{ini}}}\right) \gtrsim 55$$

FLUCTUATIONS OF MICRO-PHYSICAL QUANTUM ORIGIN

$\mathcal{H}^{-1} = (aH)^{-1}$

Comoving Hubble radius



MECHANICS OF INFLATION: CURRENT PARADIGM

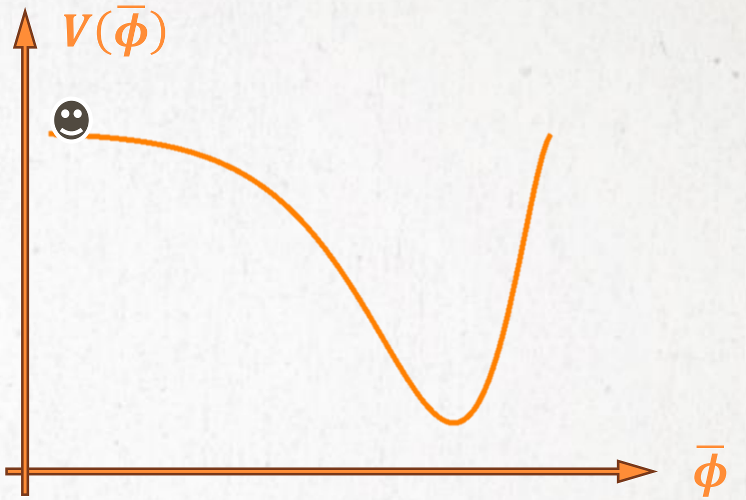
A single scalar field in slow roll does the job for both:

- The classical background…

… provided the scalar potential is flat and inflation lasts long enough

- The quantum fluctuations…

… if they emerge from the Bunch-Davies (BD) vacuum



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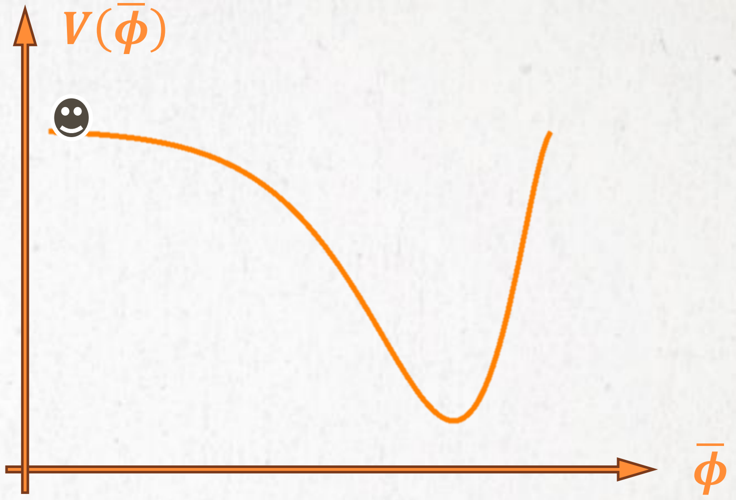

$$\phi(\vec{x},t) = \bar{\phi}(t) + Q(\vec{x},t) \qquad \text{with } Q(\vec{x},t) \ll \bar{\phi}(t)$$

Homogeneous background, slow roll: $\dot{\phi} \simeq -\frac{V_{,\phi}(\bar{\phi})}{3H} \Rightarrow H(\bar{\phi})^2 \simeq \frac{V(\bar{\phi})}{3M_{\rm Pl}^2}$ **CLASSICAL**

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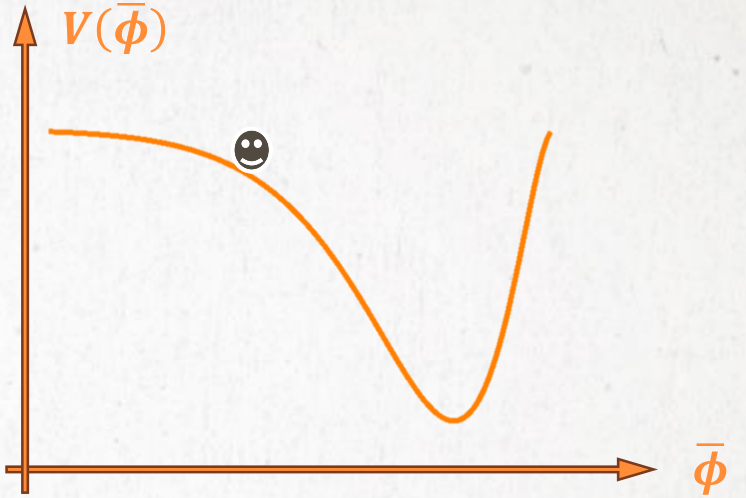

Massless, BD: $Q_k(\tau) \simeq \frac{e^{-ik\tau}}{a\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \Rightarrow \zeta_k(\tau) \simeq \frac{H}{\dot{\phi}} Q_k(\tau)$

QUANTUM

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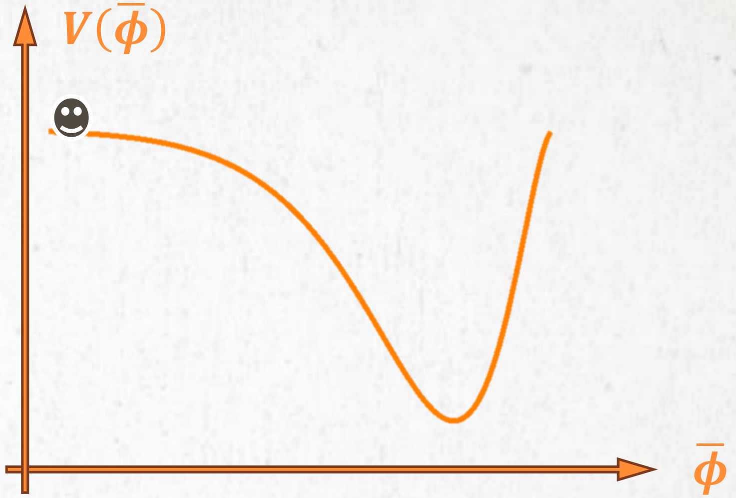


$$\zeta_k(\tau) \simeq \frac{H}{\dot{\phi}} Q_k(\tau)$$

$$\phi(\vec{x}, t) = \bar{\phi}(t) + Q(\vec{x}, t)$$

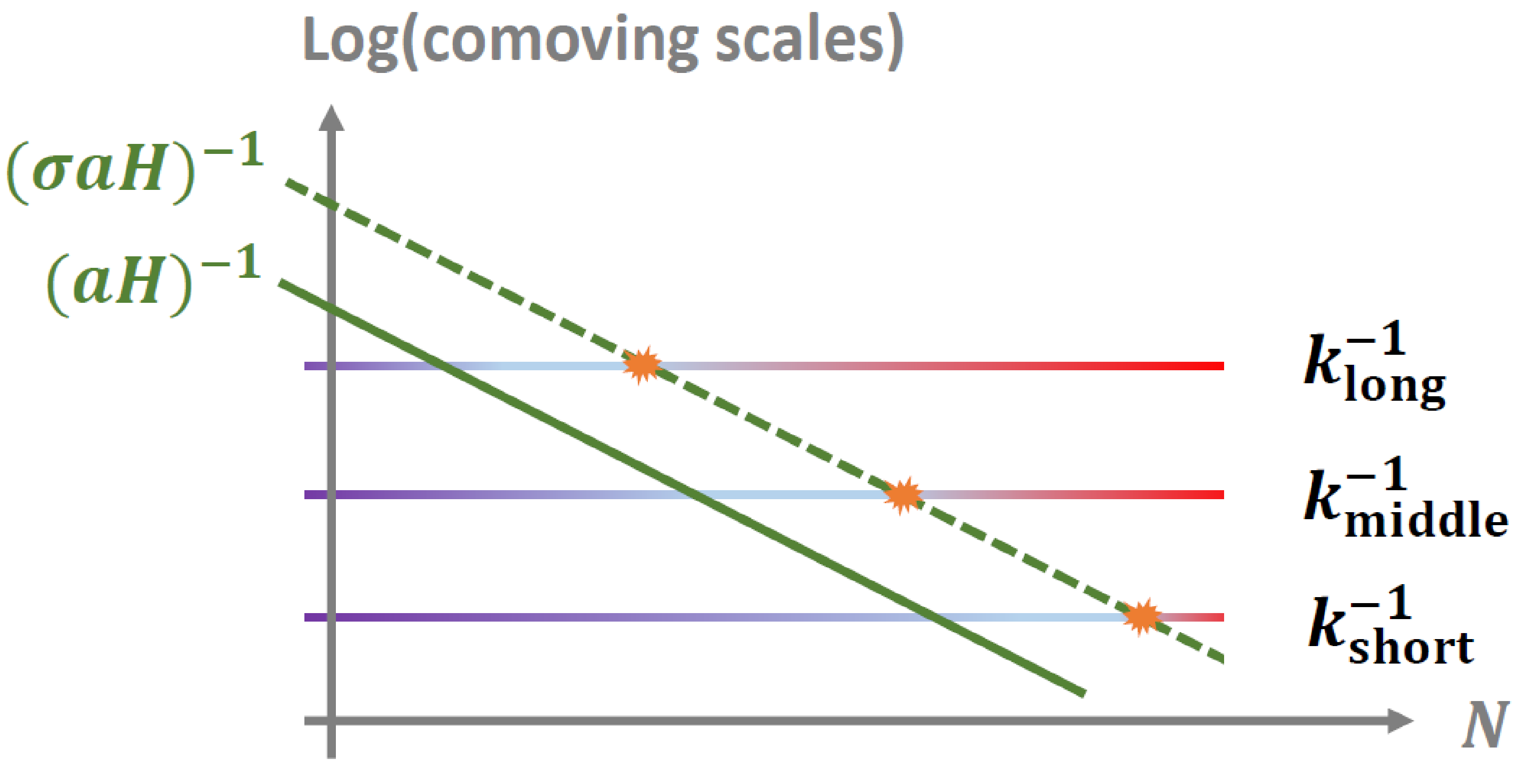
Almost scale-invariant power spectrum: $n_s \simeq 1$

Homogeneous background, slow roll: $\dot{\phi} \simeq -\frac{V_{,\phi}(\bar{\phi})}{3H}$

$$\mathcal{P}_\zeta(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$$

II. STOCHASTIC INFLATION: CONCEPTS

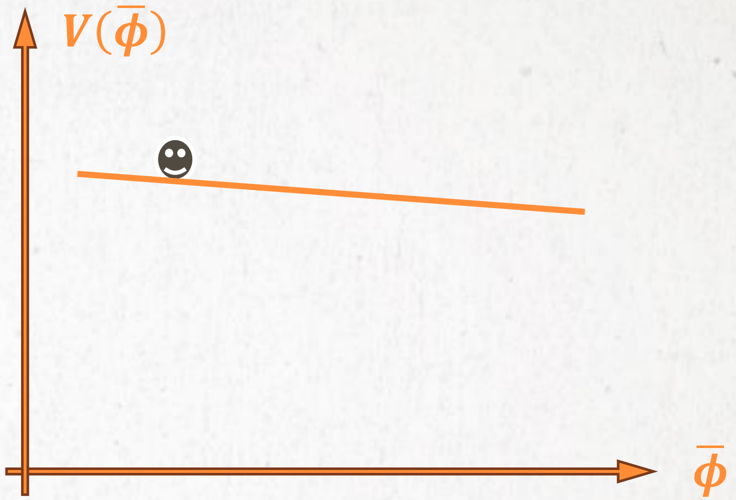
Coarse-graining the EoM…
… and applications



ACCUMULATION OF FLUCTUATIONS

With a very flat potential:

- Quantum kicks can dominate the force derived from the potential
- Even if quantum(t) $\ll$ classical(t), quantum effects can accumulate and backreact on the large-scale dynamics



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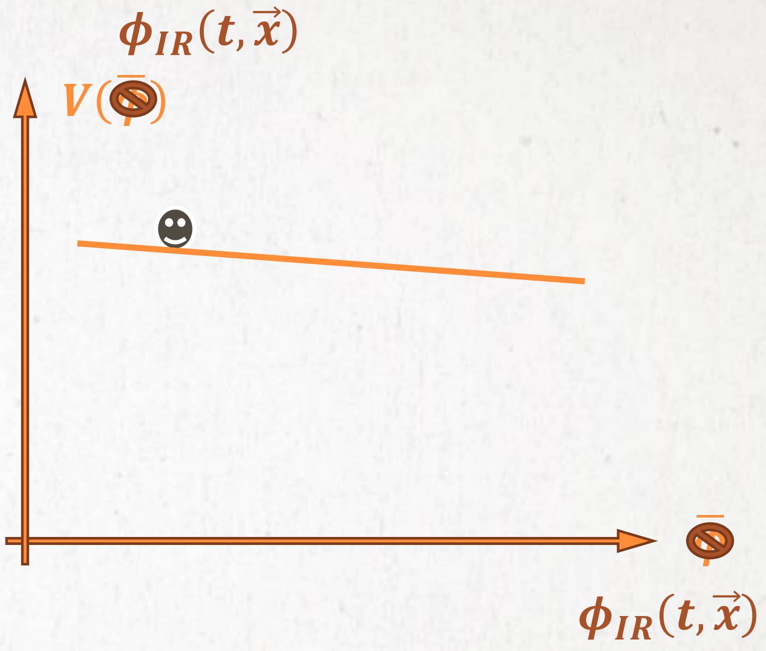

IR = Infra-Red (for large physical scales)

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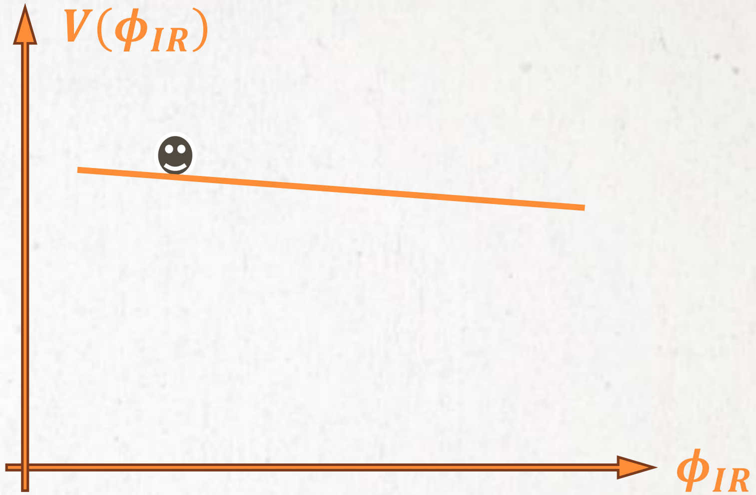


$$\boldsymbol{\phi(t,\vec{x}) = \phi_{IR}(t,\vec{x}) + \phi_{UV}(t,\vec{x})}$$

With $\phi_{UV} \ll \phi_{IR}$ …

… but possibly large inhomogeneities on large scales

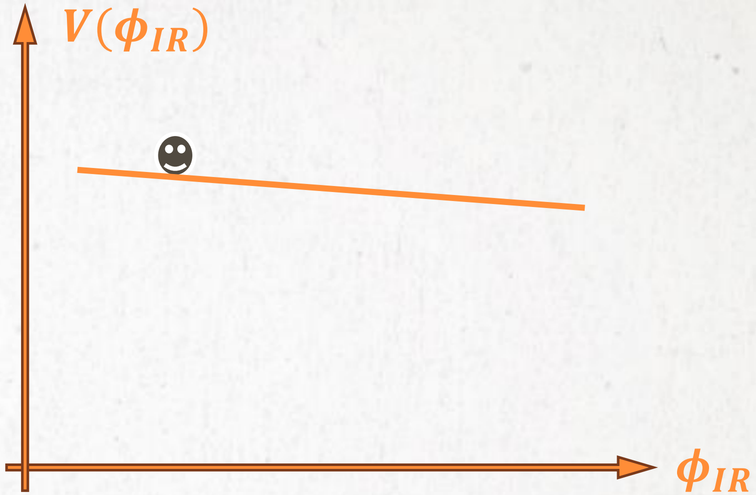
Remember that in the standard approach $\phi(t,\vec{x}) = \bar{\phi}(t) + Q(t,\vec{x})$ *and all inhomogeneities are in* $Q \ll \bar{\phi}$

WHAT ARE WE LOOKING FOR?

A theory that:

- Takes into account the effect of small-scales quantum fluctuations to describe the classical large-scale dynamics

- Arises from a perturbative expansion in ϕ_{UV} but is fully non-perturbative in ϕ_{IR}



- Should describe a dynamics with drift + diffusion = Brownian motion in a potential

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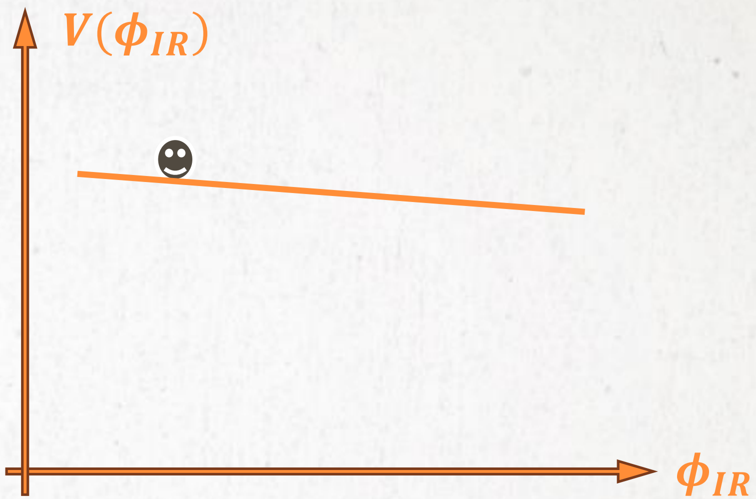

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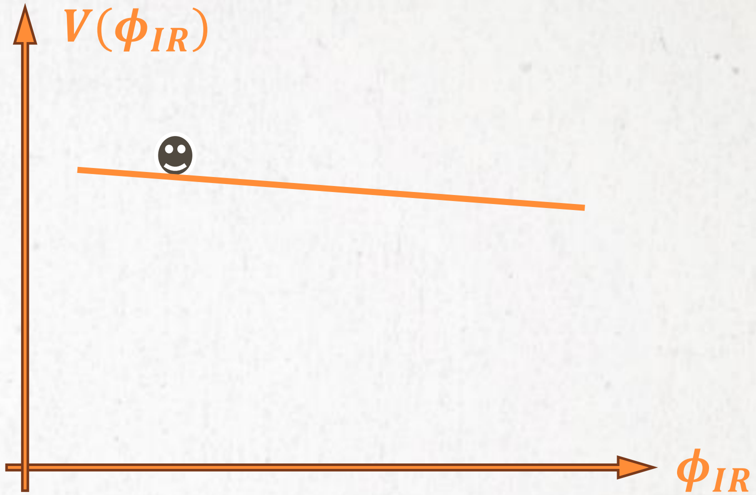
→ $$\boldsymbol{\phi(t,\vec{x}) = \phi_{IR}(t,\vec{x}) + \phi_{UV}(t,\vec{x})}$$

With $\phi_{UV} \ll \phi_{IR}$ and $\partial_x \phi_{IR} \ll \partial_t \phi_{IR}$



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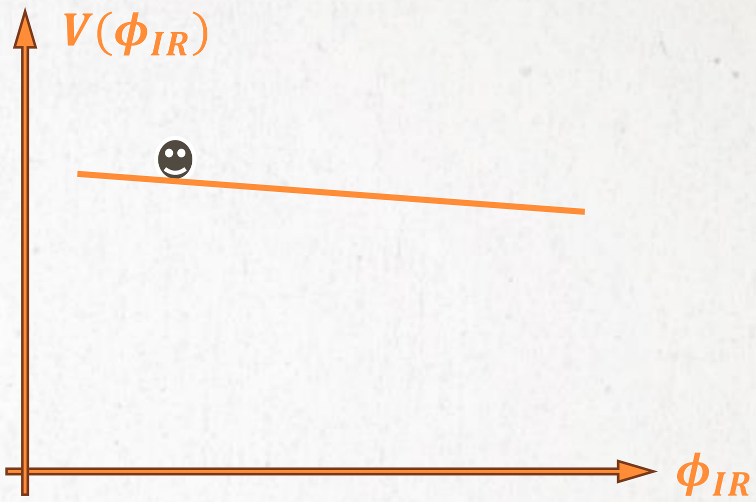

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- Should describe a dynamics with drift + diffusion = Brownian motion in a potential

  → **Langevin equation for ϕ_{IR}**

  Fokker-Planck equation for $P(\phi_{IR}, t)$

WHY STOCHASTIC?

$\phi_{IR}(t,\vec{x})$ depends on the past realisations of $\phi_{UV}(t,\vec{x})$

QUANTUM

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- Should describe a dynamics with drift + diffusion = **Brownian motion** in a potential

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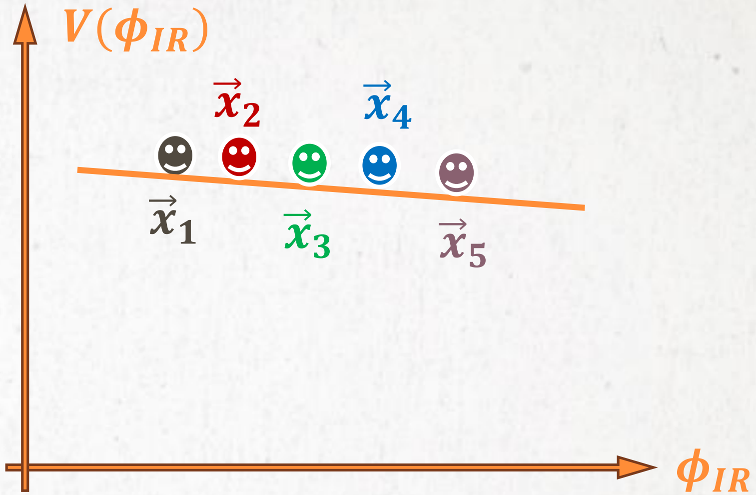
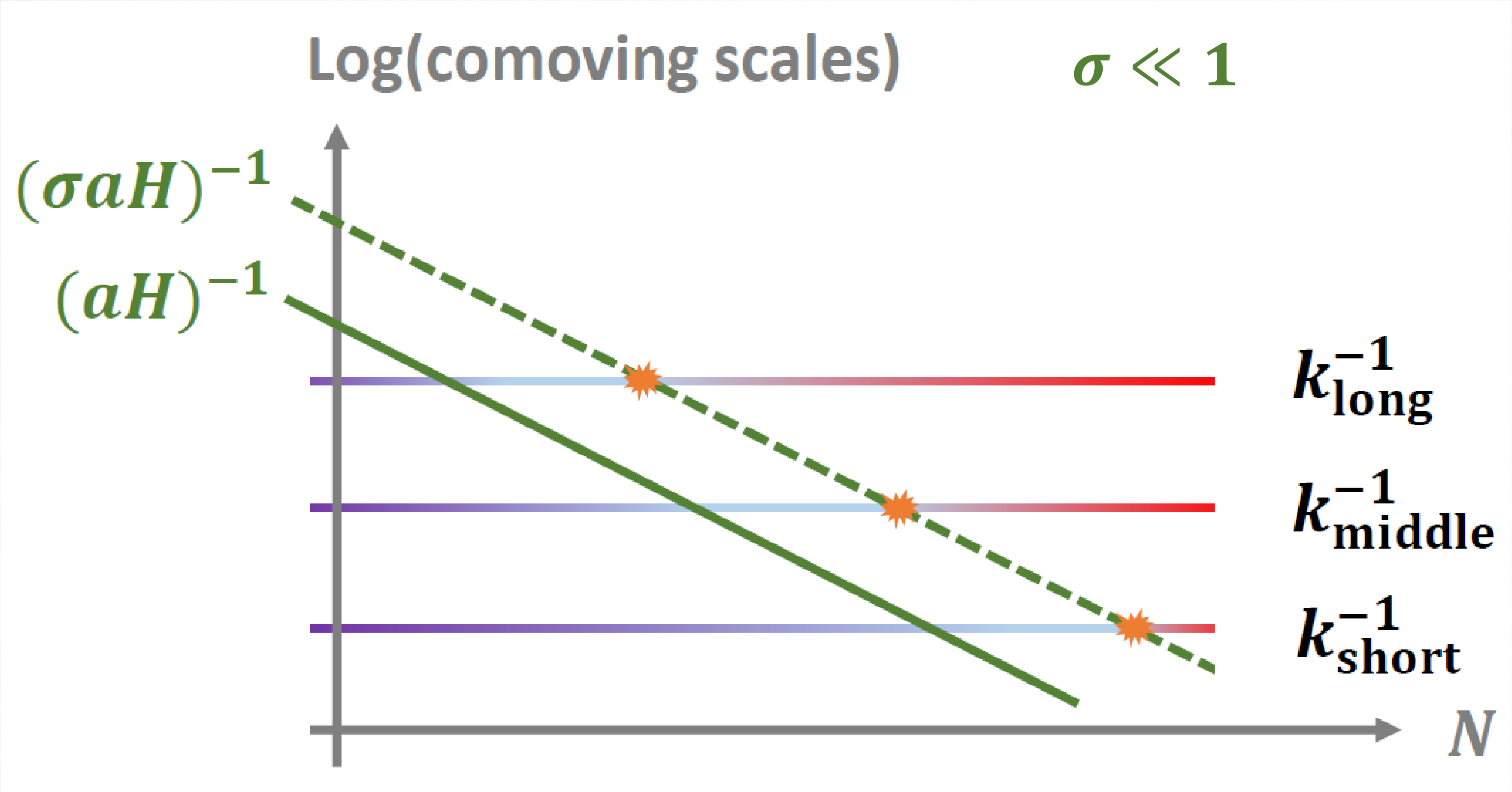
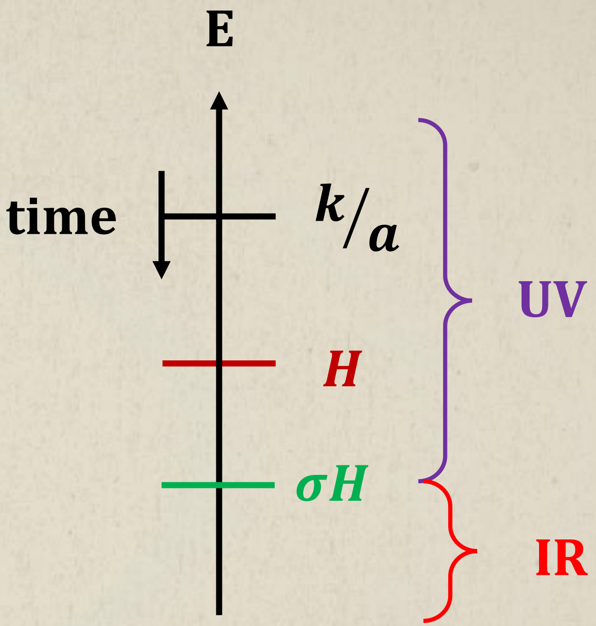


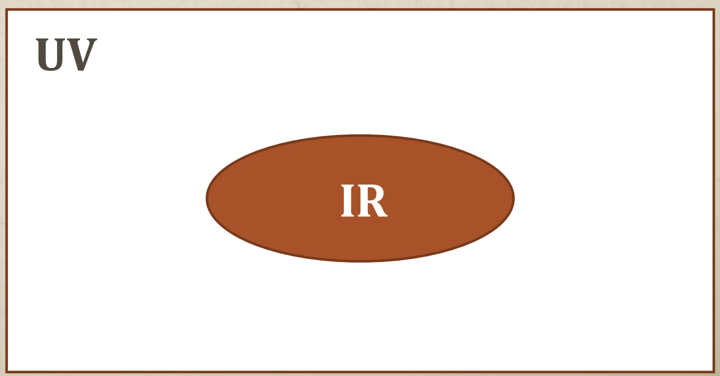
COARSE-GRAINING



- ➢ Cut-off $(\sigma a \mathrm{H})^{-1}$ defines the UV and IR sectors
- ➢ Because of time-dependence, UV modes join the IR sector
- ➢ Open, out-of-equilibrium system



With $\sigma \ll 1$



IN THE EOM

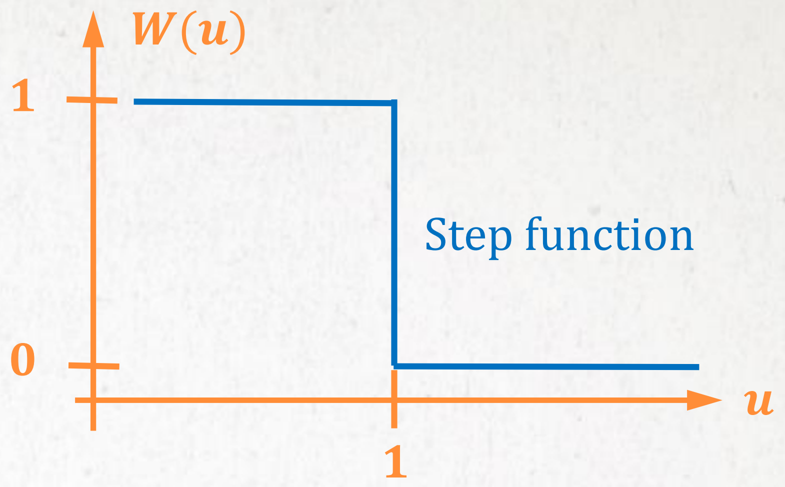
N is the number of e-folds time variable:

$$a = e^N$$

- Split IR and UV: $\phi(N,\vec{x}) = \phi_{IR}(N,\vec{x}) + \phi_{UV}(N,\vec{x})$ with

$$\phi_{IR}(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} W\left(\frac{k}{k_\sigma(N)}\right) \phi_{\vec{k}}(N)$$

Time-dependent window function that selects only $k < \boldsymbol{k_\sigma(N) = \sigma a H}$



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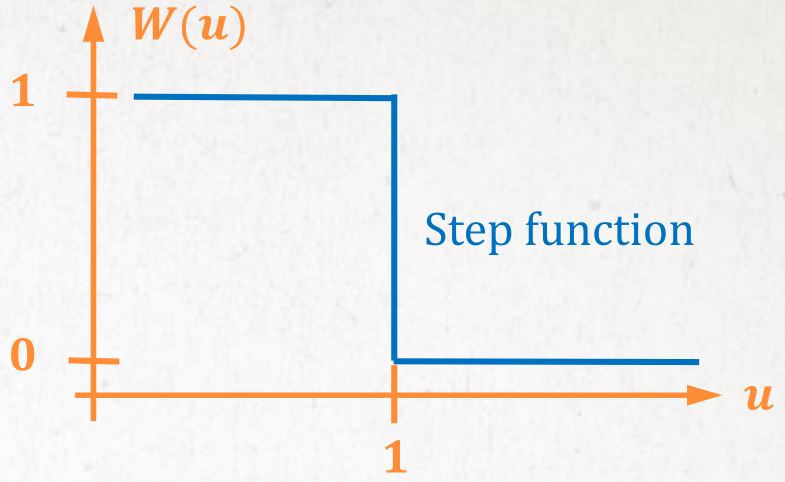

- Write the **non-linear** EoM for ϕ_{IR} in terms of the **linear** one for $\phi_{UV} \ll \phi_{IR}$:

(Slow-roll): $$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} +$$

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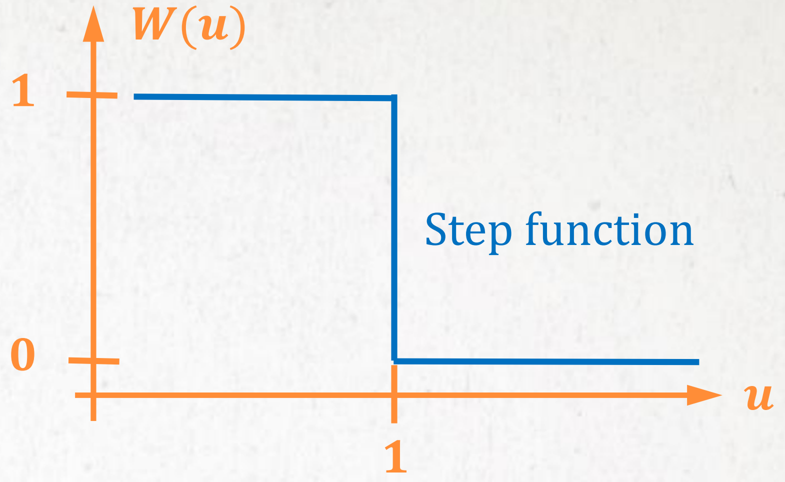

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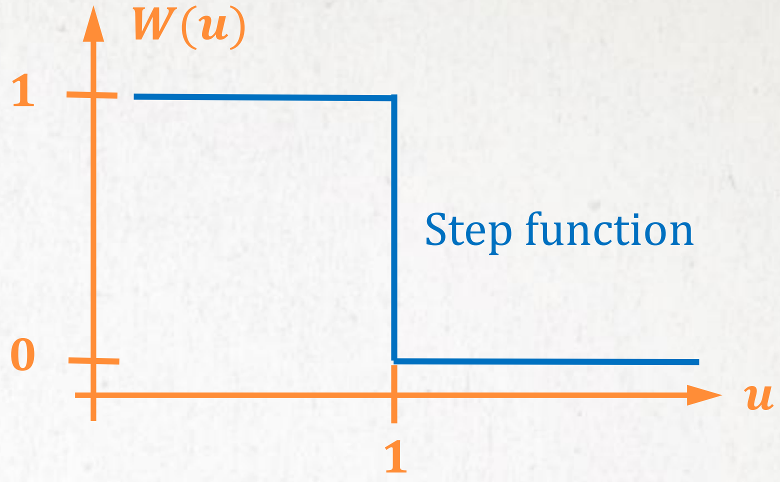

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Usual equation of motion for linear perturbations in slow-roll, supposed to still hold (IR does not flow into the UV)

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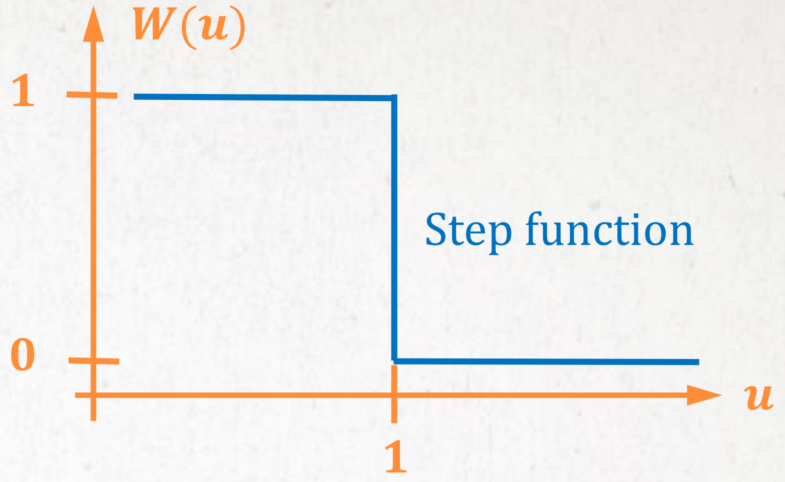

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$$+ \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} \frac{\mathrm{d}}{\mathrm{d}N}\left[W\left(\frac{k}{k_\sigma(N)}\right)\right] \phi_{\vec{k}}(N)$$

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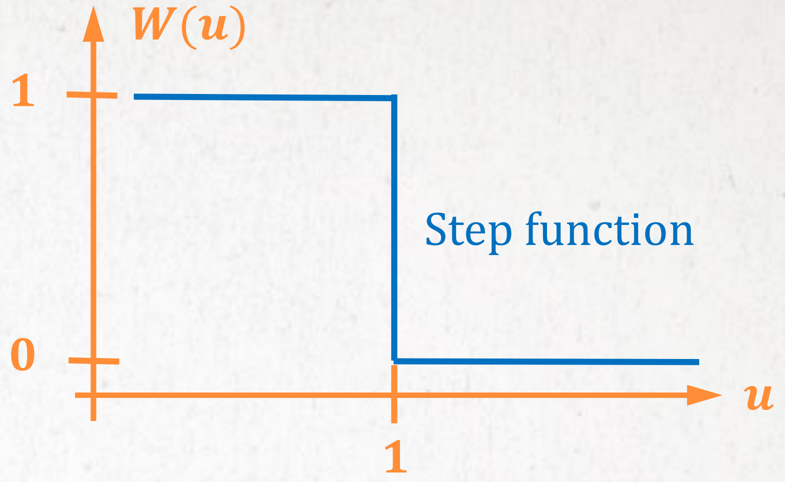

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$$+ \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} \frac{\mathrm{d}}{\mathrm{d}N}\left[W\left(\frac{k}{k_\sigma(N)}\right)\right]\phi_{\vec{k}}(N) \longrightarrow \xi_\phi(N,\vec{x}) \propto \phi_{k_\sigma(N)}(N)$$

$$\propto \delta\left(k - k_\sigma(N)\right)$$

THE LANGEVIN EQUATION

$$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} + \xi_\phi(N,\vec{x}) \text{ with}$$

$$\xi_\phi(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} \frac{\mathrm{d}}{\mathrm{d}N}\left[W\left(\frac{k}{k_\sigma(N)}\right)\right] \hat{\phi}_{\vec{k}}(N)$$

THE LANGEVIN EQUATION

A priori ξ_ϕ is a quantum operator too

↑

Fourier modes are quantised during inflation!

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$$\hat{\phi}_{\vec{k}}(N) = \phi_k(N)\hat{a}_{\vec{k}} + \phi_k^*(N)\hat{a}^\dagger_{-\vec{k}}$$

Mode decomposition on the creation-annihilation operators

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Only modes $\vec{k}$ with $|\vec{k}| = k_\sigma(N) \ll aH$ that are well above the horizon: **classicalisation**

$$\hat{\phi}_{\vec{k}}(N) = \phi_{k_\sigma}(N)\hat{a}_{\vec{k}} + \phi^*_{k_\sigma}(N)\hat{a}^\dagger_{-\vec{k}}$$

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$$\hat{\phi}_{\vec{k}}(N) = \phi_{k_\sigma}(N)\hat{a}_{\vec{k}} + \underbrace{\phi^*_{k_\sigma}(N)}_{= \phi_{k_\sigma}(N) \text{ because } k_\sigma(N) \ll aH}\hat{a}^\dagger_{-\vec{k}}$$

$aH \leftarrow -1/\tau$

Remember the massless mode functions $\quad Q_k(\tau) \simeq \frac{e^{-ik\tau}}{a\sqrt{2k}}\left(i + \frac{1}{k\tau}\right) \xrightarrow[k\tau\to 0]{} -\frac{H}{\sqrt{2k^3}}$

THE LANGEVIN EQUATION

A priori ξ_ϕ is a quantum operator too

↑

Fourier modes are quantised during inflation!

$$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} + \xi_\phi(N,\vec{x}) \text{ with}$$

$$\xi_\phi(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} \frac{\mathrm{d}}{\mathrm{d}N}\left[W\left(\frac{k}{k_\sigma(N)}\right)\right] \hat{\phi}_{\vec{k}}(N)$$

Only modes $\vec{k}$ with $|\vec{k}| = k_\sigma(N) \ll aH$ that are well above the horizon: **classicalisation**

$$\hat{\phi}_{\vec{k}}(N) = \phi_{k_\sigma}(N)\hat{a}_{\vec{k}} + \phi^*_{k_\sigma}(N)\hat{a}^\dagger_{-\vec{k}} = \phi_{k_\sigma}(N)\underbrace{\left(\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\right)}$$

$b_{\vec{k}}$: only « quantum » operator

Classical random variable

A CLASSICAL NOISE WITH QUANTUM STATISTICS

- $$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} + \xi_\phi(N,\vec{x}) \text{ with}$$

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Classical Gaussian random variable

$$\langle \boldsymbol{b}_{\vec{k}} \rangle = \mathbf{0}\,;\langle \boldsymbol{b}_{\vec{k}}\boldsymbol{b}_{\vec{k}'} \rangle = (\mathbf{2}\boldsymbol{\pi})^{\mathbf{3}}\boldsymbol{\delta}^{(\mathbf{3})}(\vec{\boldsymbol{k}}+\vec{\boldsymbol{k}}')$$

A CLASSICAL NOISE WITH QUANTUM STATISTICS

- $$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} + \xi_\phi(N,\vec{x}) \text{ with}$$

$\langle b_{\vec{k}}\rangle = 0\,;\langle b_{\vec{k}} b_{\vec{k}'}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')$

$$\xi_\phi(N,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\,\vec{k}\cdot\vec{x}} \frac{\mathrm{d}}{\mathrm{d}N}\left[W\left(\frac{k}{k_\sigma(N)}\right)\right]\phi_{k_\sigma}(N)\boldsymbol{b}_{\vec{k}}$$

Spatial correlation: $r = |\vec{x}-\vec{x}'|$

- $$\langle\xi_\phi(N,\vec{x})\rangle = 0\,;\ \langle\xi_\phi(N,\vec{x})\xi_\phi(N',\vec{x}')\rangle = \delta(N-N')\mathrm{sinc}(k_\sigma r)\mathcal{P}_\phi(N,k_\sigma(N))$$

White noise

Power spectrum of linear fluctuations at the scale $k_\sigma(N)$

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White noise

White noise is uncorrelated in time…

… but a smoother window function gives coloured noise!

A CLASSICAL NOISE WITH QUANTUM STATISTICS

- $\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H^2} + \xi_\phi(N,\vec{x})$ with

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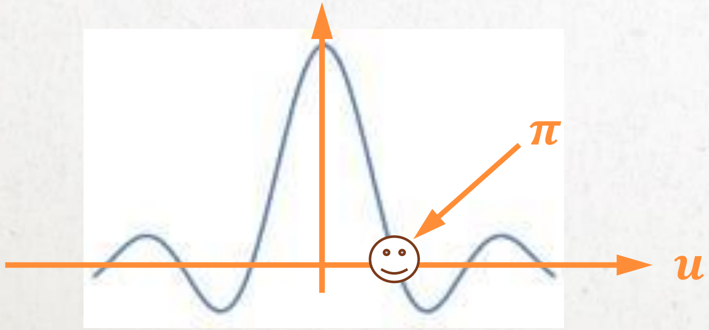
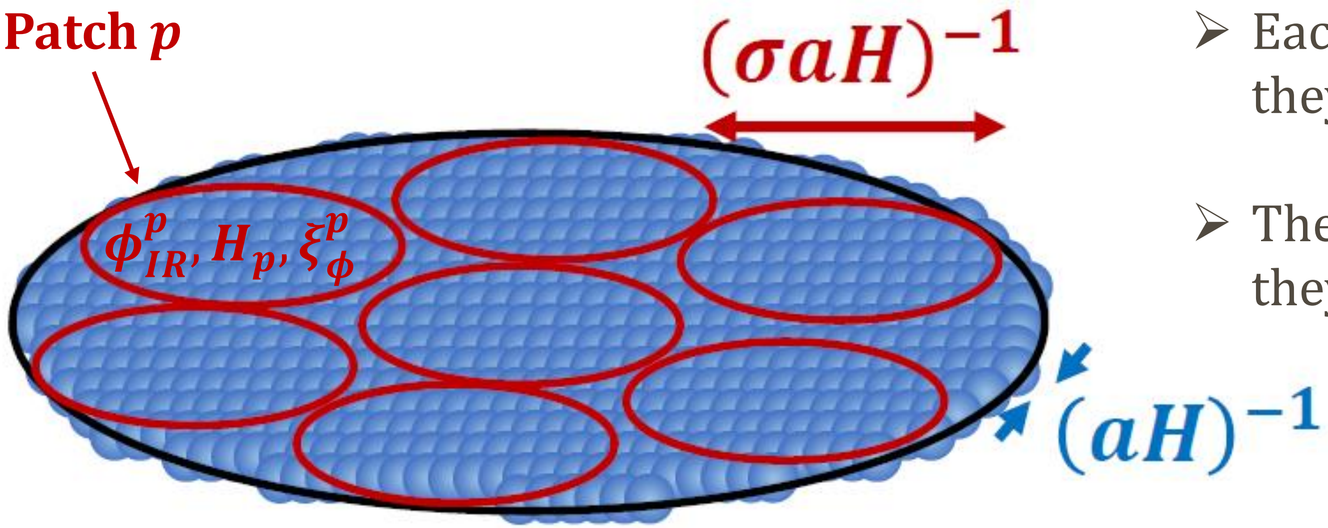
$\mathrm{sinc}(u) = \sin(u)/u$



$\mathrm{sinc}(u) \simeq 1$ for $u \ll 1$ Maximally correlated for $r \ll k_\sigma^{-1}$

$\mathrm{sinc}(u) \simeq 0$ for $u \gg 1$ Uncorrelated for $r \gg k_\sigma^{-1}$

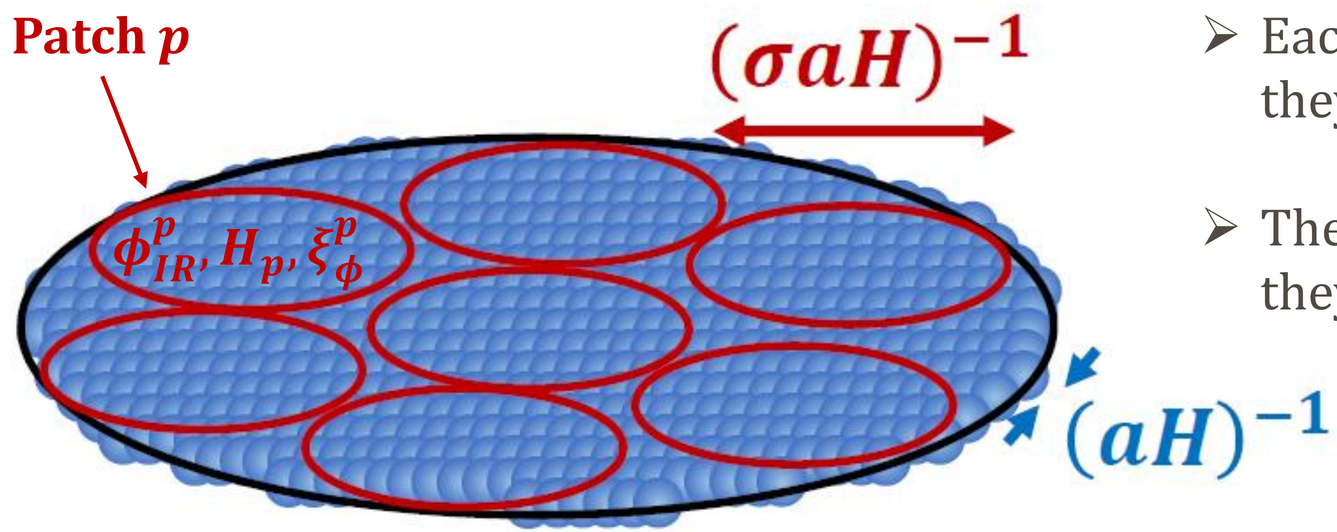
THE SEPARATE UNIVERSE APPROACH



- Each **σ-Hubble patch** evolves independently: they are separated by $r \gg k_\sigma^{-1}$
- The noise takes a unique value inside each patch: they can only be separated by $r \ll k_\sigma^{-1}$

  → Focus on 1-pt statistics

THE SEPARATE UNIVERSE APPROACH



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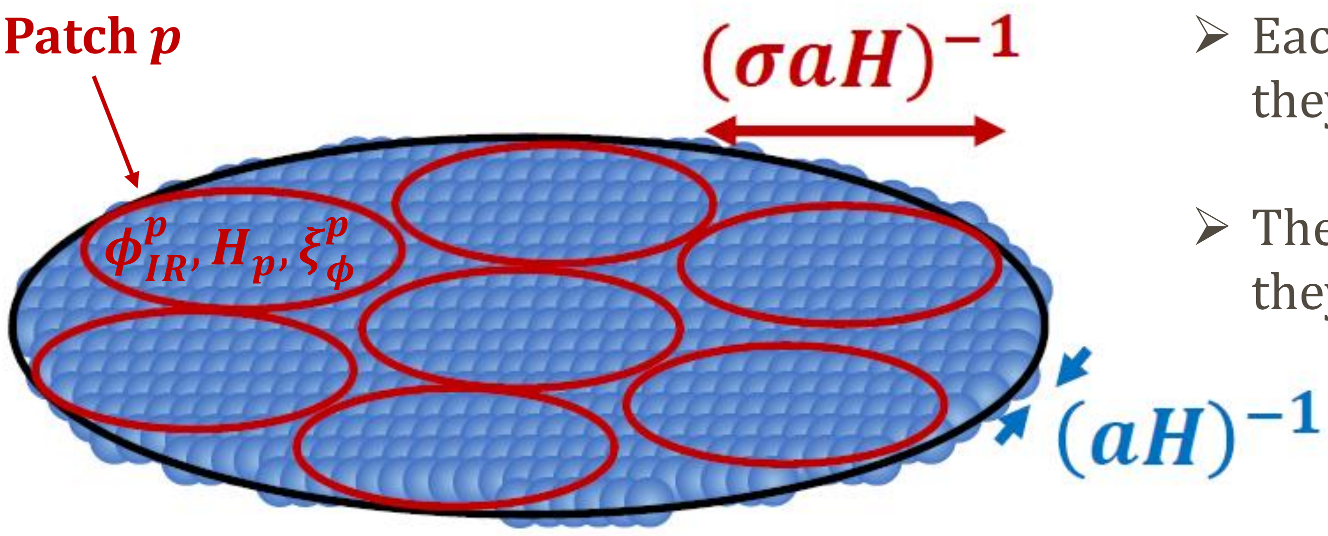
  → Focus on 1-pt statistics

- For each patch p, $\frac{\mathrm{d}\phi_{IR}^{p}}{\mathrm{d}N} = -\frac{V_{,\phi}\left(\phi_{IR}^{p}\right)}{3H_p^2} + \xi_\phi^p(N)$, with $\langle \xi_\phi^p(N)\xi_\phi^q(N')\rangle = \delta(N-N')\delta^{pq}\mathcal{P}_{\phi^p}\left(N, k_\sigma^p(N)\right)$

  Note that the time variable N was chosen such that it is shared by all patches → uniform-N gauge ~ flat gauge

$$\phi_{IR}(N,\vec{x}) \rightarrow \phi_{IR}^{p}(N)$$

THE SEPARATE UNIVERSE APPROACH



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  Note that the time variable N was chosen such that it is shared by all patches → uniform-N gauge ~ flat gauge

- Large-scales fluctuations are found by comparing ζ_p in different patches , with

  δN-formalism: $\delta N_p = N_p - \bar{N} = -\zeta_p$, where N_p is the time needed to reach the final hypersurface

[Wands, Malik, Lyth, Riddle 2000] [Lyth, Malik, Sasaki 2005] [Fujita, Kawasaki, Tada 2014]

A CLASSICAL NOISE WITH QUANTUM STATISTICS

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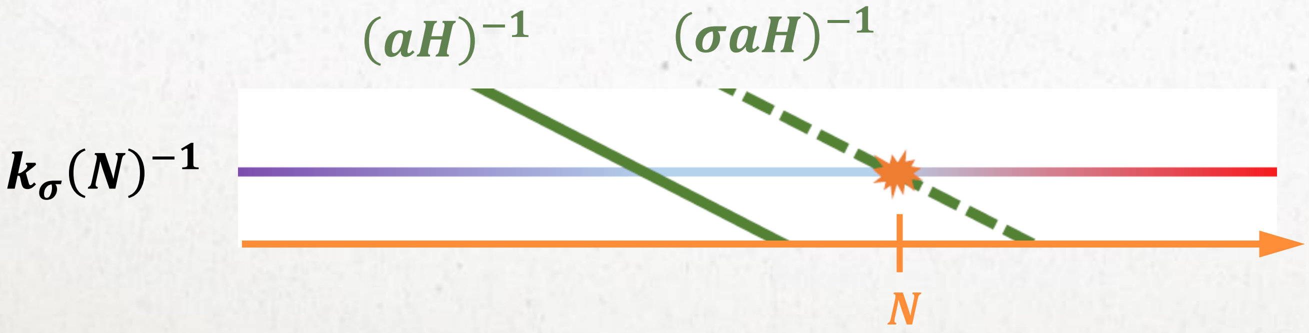
- $\langle \xi_\phi(N,\vec{x}) \rangle = 0\,;\ \langle \xi_\phi(N,\vec{x})\xi_\phi(N',\vec{x}') \rangle = \delta(N-N')\mathrm{sinc}(k_\sigma r)\mathcal{P}_\phi\big(N, k_\sigma(N)\big)$

Power spectrum of linear fluctuations at the scale $k_\sigma(N)$



At the time N, the quantum UV mode $k_\sigma(N)$ joins the IR sector and kicks it according to its power spectrum

TO BE MARKOVIAN OR NOT TO BE

- To know the noise statistics you should know $\mathcal{P}_\phi\left(N, k_\sigma(N)\right)$

- The quantum UV modes verify the usual linear EoM

$$\ddot{\phi}_{\vec{k}} + 3H\dot{\phi}_{\vec{k}} + \left(\frac{k^2}{a^2} + m_{\mathrm{eff}}^2\right)\phi_{\vec{k}} = 0$$

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- But… $H(\phi_{IR})$and $m_{\text{eff}}^2(\phi_{IR})$ depend on the value of $\phi_{IR} \rightarrow$ stochastic!

Ex: Friedmann equation: $3H^2(\phi_{IR})M_{\text{Pl}}^2 = V(\phi_{IR}) + \frac{1}{2}\dot{\phi}_{IR}^2$

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Ex: Friedmann equation: $3H^2 M_{\text{Pl}}^2 = V(\phi_{IR}) + \frac{1}{2}\dot{\phi}_{IR}^2$

- In principle, you can not solve the two systems separately

(actually, one IR system and as many UV ones as k modes / time steps)

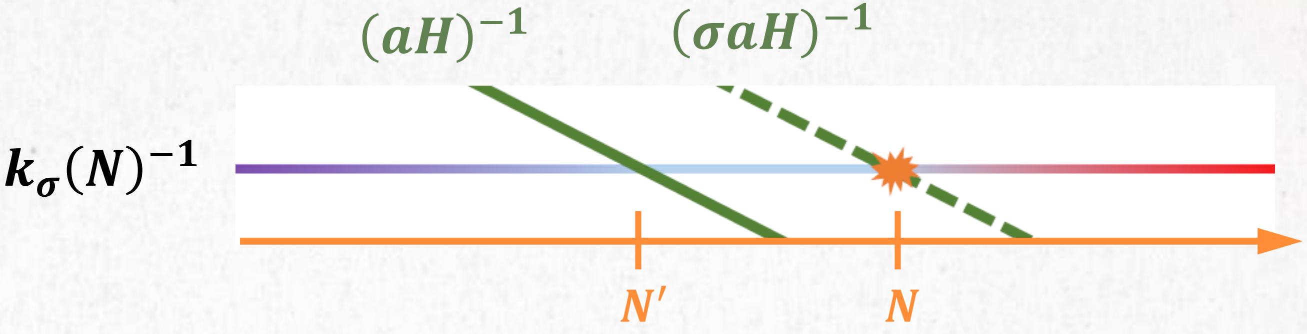
- The system is **non-Markovian**: The noise statistics depends on its own past realisations

(to find $\phi_{k_{\sigma}}(N)$ you need to know $\phi_{IR}(N'), \forall\, N' < N$)

PARTICULAR CASES

- Massless scalar field: $\mathcal{P}_\phi(N, k_\sigma(N)) = \left(\frac{H(N')}{2\pi}\right)^2$ with $N' < N$ the time of horizon crossing

Non-Markovian



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- Massless + slow-variation: $H(N') = H(N) \rightarrow$ Markovian but **multiplicative** noise

$$\langle \xi_\phi^2 \rangle \propto \left(\frac{H[\phi_{IR}(N)]}{2\pi}\right)^2$$

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- Massless + slow-variation + spectator: $H(N') = H(N)$ independent on ϕ_{IR} $\rightarrow$ **Additive** noise

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- Massless + de Sitter (implying spectator): $H(N') = H_0$ a **constant**

$$\langle \xi_\phi^2 \rangle \propto \left(\frac{H_0}{2\pi}\right)^2$$

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In the following, we assume massless + slow-variation → Markovian but multiplicative

FOKKER-PLANCK EQUATION

Normalised, centered Gaussian variable

Focus on 1 patch: $$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H_p^2} + \frac{H[\phi_{IR}(N)]}{2\pi} \times \xi(N)\,, \qquad \langle \xi(N)\xi(N')\rangle = \delta(N-N')$$

Square-root of the noise amplitude

FOKKER-PLANCK EQUATION

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From stochastic differential equation for ϕ_{IR} to partial differential equation for its PDF $P(\phi_{IR}, N)$:

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From stochastic differential equation for ϕ_{IR} to partial differential equation for its PDF $P(\phi_{IR}, N)$:

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}} \left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H(\phi_{IR})^2} - \boldsymbol{\alpha} \frac{H(\phi_{IR})}{2\pi} \frac{\partial}{\partial \phi_{IR}} \frac{H(\phi_{IR})}{2\pi} \right) \boldsymbol{P} \right] + \frac{1}{2} \frac{\partial^2}{\partial \phi_{IR}^2} \left[\left(\frac{H(\phi_{IR})}{2\pi} \right)^2 \boldsymbol{P} \right]$$

Convection-diffusion equation for P

Classical drift — Noise-induced drift — Diffusion of quantum origine

Total drift

FOKKER-PLANCK EQUATION

$$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = -\frac{V_{,\phi}(\phi_{IR})}{3H_p^2} + \frac{H[\phi_{IR}(N)]}{2\pi} \times \xi(N)\,, \qquad \langle \xi(N)\xi(N') \rangle = \delta(N - N')$$

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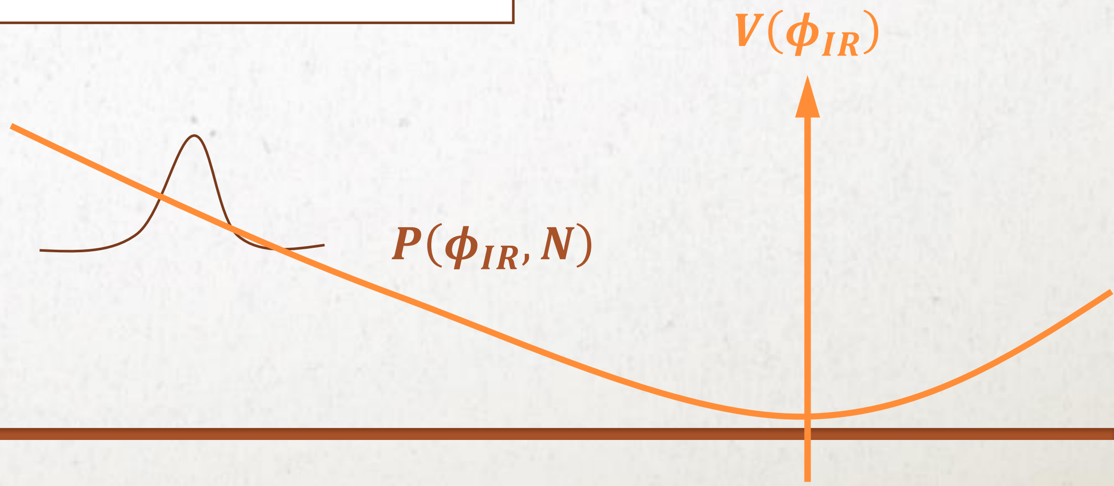
Convection-diffusion equation for P

Convection: $\rightarrow$

Diffusion: $\leftrightarrow$



STOCHASTIC DIFFERENTIAL EQUATIONS
AND DISCRETISATION SCHEMES

At the discrete level: $\Delta W_n \Delta W_m = \delta_{nm} \Delta N$

Langevin equation: $$\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = h[\phi_{IR}(N)] + g\big[\phi_{IR}(N)\big] \times \xi(N)\,, \qquad \langle \xi(N)\xi(N') \rangle = \delta(N - N')$$

How to solve it?!

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How to solve it?!

Numerically, need to specify the discretisation:

$$\phi_{IR}^{n+1} - \phi_{IR}^{n} = h[\phi_{IR}^{n+\alpha}]\Delta N + g[\phi_{IR}^{n+\alpha}] \times \Delta W_n$$

$\alpha \in [0,1]$ parameterises the time $N_{n+\alpha} = N_n + \alpha \Delta N$ at which the RHS is evaluated

- Irrelevant for deterministic differential equations (except stability, conservativity, etc.)
- Crucial for stochastic differential equations (physical result depends on it)

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Langevin equation: $\frac{\mathrm{d}\phi_{IR}}{\mathrm{d}N} = h[\phi_{IR}(N)] + g[\phi_{IR}(N)] \times \xi(N)$, $\langle \xi(N)\xi(N') \rangle = \delta(N - N')$

How to solve it?!

Numerically, need to specify the discretisation:

$$\underbrace{\phi_{IR}^{n+1} - \phi_{IR}^{n}}_{\Delta\phi_{IR} = O(\Delta N^{1/2})} = \underbrace{h[\phi_{IR}^{n+\alpha}]\Delta N}_{O(\Delta N)} + \underbrace{g[\phi_{IR}^{n+\alpha}] \times \Delta W_n}_{O(\Delta N^{1/2})}$$

Mathematical fact: Brownian motion is continuous but not differentiable, $\frac{\Delta W_n}{\Delta N} \xrightarrow[\Delta N \to 0]{} \infty$

STOCHASTIC DIFFERENTIAL EQUATIONS
AND DISCRETISATION SCHEMES

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like ODE: does not matter

STOCHASTIC DIFFERENTIAL EQUATIONS
AND DISCRETISATION SCHEMES

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different from ODE: does matter!

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- $$g[\phi_{IR}^{n+\alpha}]\Delta W_n = g[\phi_{IR}^{n}]\Delta W_n + \boldsymbol{\alpha} \frac{\partial g}{\partial \phi_{IR}}[\phi_{IR}^{n}]\Delta\phi_{IR}\Delta W_n$$ $$= g[\phi_{IR}^{n}]\Delta W_n + \boldsymbol{\alpha} g[\phi_{IR}^{n}] \frac{\partial g}{\partial \phi_{IR}}[\phi_{IR}^{n}]\Delta N$$

Noise-induced drift $\propto \Delta N$

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H(\phi_{IR})^2} - \boldsymbol{\alpha}\frac{H(\phi_{IR})}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H(\phi_{IR})}{2\pi}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H(\phi_{IR})}{2\pi}\right)^2 \boldsymbol{P}\right]$$

Inflationary universe

Formal QFT results in de Sitter (spectator field)

FOKKER-PLANCK EQUATION: APPLICATIONS

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Inflationary universe

- $\frac{\partial P_{\mathrm{eq}}}{\partial N} = 0 \Rightarrow P_{\mathrm{eq}} = \frac{P_{0,\alpha}}{[v(\phi_{IR})]^{1-\alpha}} e^{\frac{1}{v(\phi_{IR})}}$, $v = \frac{V}{M_{\mathrm{Pl}}^4} \ll 1$ to work in the perturbative regime of quantum gravity

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Inflationary universe

- $\frac{\partial P_{\rm eq}}{\partial N} = 0 \Rightarrow P_{\rm eq} = \frac{P_{0,\alpha}}{[v(\phi_{IR})]^{1-\alpha}} e^{\frac{1}{v(\phi_{IR})}}, \quad v = \frac{V}{M_{\rm Pl}^4} \ll 1$ to work in the perturbative regime of quantum gravity

- $\langle N \rangle \simeq \underbrace{\frac{1}{M_{\rm Pl}^2}\int_{\phi_{IR}^{\rm end}}^{\phi_{IR}^{\rm ini}} \mathrm{d}\phi_{IR} \frac{v(\phi_{IR})}{v_{,\phi}(\phi_{IR})}}_{N_{\rm cl}}[1 + (1+\alpha)v - \eta_{\rm cl} + \cdots], \quad \eta_{\rm cl} = \frac{v v_{,\phi\phi}}{v_{,\phi}^2} \ll 1$

may be large in a numerical resolution
but was supposed small in this analytical result

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H(\phi_{IR})^2} - \boldsymbol{\alpha}\frac{H(\phi_{IR})}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H(\phi_{IR})}{2\pi}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H(\phi_{IR})}{2\pi}\right)^2 \boldsymbol{P}\right]$$

Inflationary universe

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- $P_\zeta \sim \frac{{\rm d}\langle \delta N^2 \rangle}{{\rm d}\langle N \rangle} \simeq P_\zeta^{\rm cl}[1 + (5+2\alpha)v - 4\,\eta_{\rm cl} + \cdots]$

stochastic correction to $(\boldsymbol{n_s}, \boldsymbol{r})$

[A. Starobinsky, V. Vennin 2015] *only $\alpha = 0$*

[L. Pinol, Y. Tada, S. Renaux-Petel 2018]
Class.Quant.Grav. 36 no.7, (2019) 07LT01

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H(\phi_{IR})^2} - \alpha \frac{H(\phi_{IR})}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H(\phi_{IR})}{2\pi}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H(\phi_{IR})}{2\pi}\right)^2 \boldsymbol{P}\right]$$

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- $\frac{\partial P_{\rm eq}}{\partial N} = 0 \Rightarrow P_{\rm eq} = \frac{P_{0,\alpha}}{[v(\phi_{IR})]^{1-\alpha}} e^{\frac{1}{v(\phi_{IR})}}\,, \quad v = \frac{V}{M_{\rm Pl}^4} \ll 1$ to work in the perturbative regime of quantum gravity
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stochastic correction to $(\boldsymbol{n_s}, \boldsymbol{r})$

[L. Pinol, Y. Tada, S. Renaux-Petel 2018]
Class.Quant.Grav. 36 no.7, (2019) 07LT01

α always multiplies $v \ll 1 \rightarrow$ discretisation scheme does not affect observables (also true non-perturbatively)

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H_0^2} - \alpha \frac{H_0}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H_0}{2\pi}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H_0}{2\pi}\right)^2 \boldsymbol{P}\right]$$

no ambiguity of discretisation

$H(\phi_{IR}) \to H_0$

Formal QFT results in de Sitter (spectator field)

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H_0^2} - \boldsymbol{\alpha}\cancel{\frac{H_0}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H_0}{2\pi}}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H_0}{2\pi}\right)^2 \boldsymbol{P}\right]$$

$H(\phi_{IR}) \rightarrow H_0$

Formal QFT results in de Sitter (spectator field)

- $\frac{\partial P_{\mathrm{eq}}}{\partial N} = 0 \Rightarrow P_{\mathrm{eq}} = P_0 e^{\frac{-8\pi^2 V(\phi_{IR})}{3H_0^4}}$: large deviations from gaussianity if V not quadratic → non-perturbative result

FOKKER-PLANCK EQUATION: APPLICATIONS

$$\frac{\partial \boldsymbol{P}}{\partial N} = \frac{\partial}{\partial \phi_{IR}}\left[\left(\frac{V_{,\phi}(\phi_{IR})}{3H_0^2} - \boldsymbol{\alpha}\frac{H_0}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H_0}{2\pi}\right)\boldsymbol{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi_{IR}^2}\left[\left(\frac{H_0}{2\pi}\right)^2 \boldsymbol{P}\right]$$

$H(\phi_{IR}) \to H_0$

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Without noticing, we have **resummed** IR divergencies in de Sitter: highly non-trivial! → Look at $V(\phi_{IR}) = \lambda\phi_{IR}^4$

FOKKER-PLANCK EQUATION: APPLICATIONS

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(the term $\alpha \frac{H_0}{2\pi}\frac{\partial}{\partial \phi_{IR}}\frac{H_0}{2\pi}$ is crossed out)

$H(\phi_{IR}) \to H_0$

Formal QFT results in de Sitter (spectator field)

- $\frac{\partial P_{\mathrm{eq}}}{\partial N} = 0 \Rightarrow P_{\mathrm{eq}} = P_0 e^{\frac{-8\pi^2 V(\phi_{IR})}{3H_0^4}}$: large deviations from gaussianity if V not quadratic → non-perturbative result

Without noticing, we have **resummed** IR divergencies in de Sitter: highly non-trivial! → Look at $V(\phi_{IR}) = \lambda\phi_{IR}^4$

- $\langle\phi_{IR}\rangle = 0, \qquad \langle\phi_{IR}^2\rangle = \frac{H_0^2 N}{4\pi^2}\left(1 - \frac{\lambda N^2}{6\pi^2} + O(\lambda^2 N^4)\right), \qquad \ldots$

IR (long time) divergence

1-loop QFT computation in de Sitter spacetime

[A. Starobinsky, J. Yokoyama 1994]
[N. Tsamis, R. Woodard 2005]
[F. Finelli et al 2010]
[V. Gorbenko, L. Senatore 2020]

III. STOCHASTIC INFLATION IN CURVED (PHASE) SPACE

Highlights of 2019-20

Welcome to the 2019-20 Highlights of *Classical and Quantum Gravity*. These articles are selected by the CQG Editorial Board as some of the best CQG content published in 2019 and the first half of 2020.

We hope that you will enjoy reading these papers and that you will publish your next paper with *Classical and Quantum Gravity*.

You can also view the highlights of 2017, 2016, 2015, 2014–2015, 2013–2014, 2012–2013, 2011–2012 and 2010–2011

Full 3D numerical relativity simulations of neutron star–boson star collisions with BAM
Tim Dietrich *et al* 2019 *Class. Quantum Grav.* **36** 025002
+ Open abstract View article PDF

Testing quantum black holes with gravitational waves
Valentino F Foit and Matthew Kleban 2019 *Class. Quantum Grav.* **36** 035006
+ Open abstract View article PDF

Losing the IR: a holographic framework for area theorems
Netta Engelhardt and Sebastian Fischetti 2019 *Class. Quantum Grav.* **36** 035008
+ Open abstract View article PDF

Strong cosmic censorship for charged de Sitter black holes with a charged scalar field
Oscar J C Dias *et al* 2019 *Class. Quantum Grav.* **36** 045005
+ Open abstract View article PDF

Inflationary stochastic anomalies
Lucas Pinol *et al* 2019 *Class. Quantum Grav.* **36** 07LT01
+ Open abstract View article PDF

Inflationary stochastic anomalies…

… and their resolution

[L. Pinol, Y. Tada, S. Renaux-Petel 2018]
Classical and Quantum Gravity
CQG 36 no.7, (2019) 07LT01

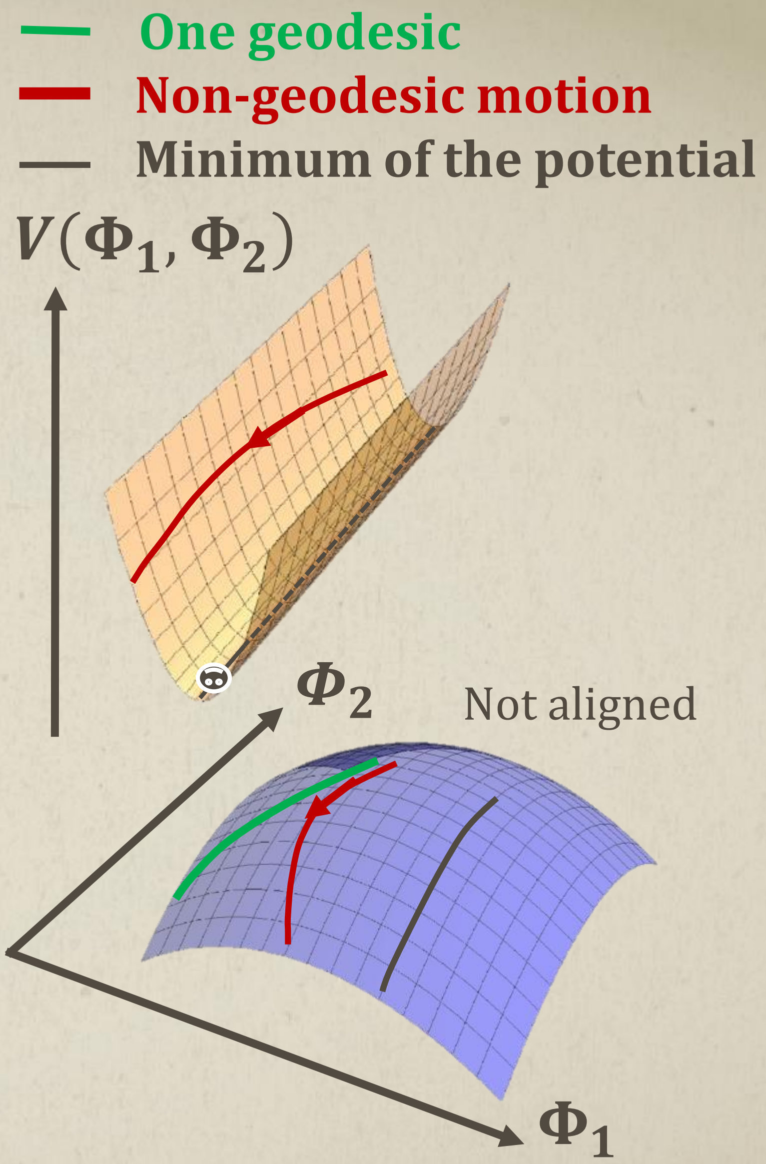
[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
Journal of Cosmology and Astroparticle Physics,
JCAP04(2021)048

MULTIFIELD INFLATION WITH CURVED FIELD SPACE

$$S = \int \sqrt{-g} \left(\frac{R}{2} - \sum_{I,J} g^{\mu\nu} G_{IJ}(\boldsymbol{\phi}) \partial_\mu \phi^I \partial_\nu \phi^J - V(\boldsymbol{\phi}) \right)$$

Kinetic couplings via $G_{IJ}(\boldsymbol{\phi})$

Non-derivative couplings via $V(\boldsymbol{\phi})$



Curved field space

Scalar curvature: $R_{fs} \neq 0$

MULTIFIELD INFLATION WITH CURVED FIELD SPACE

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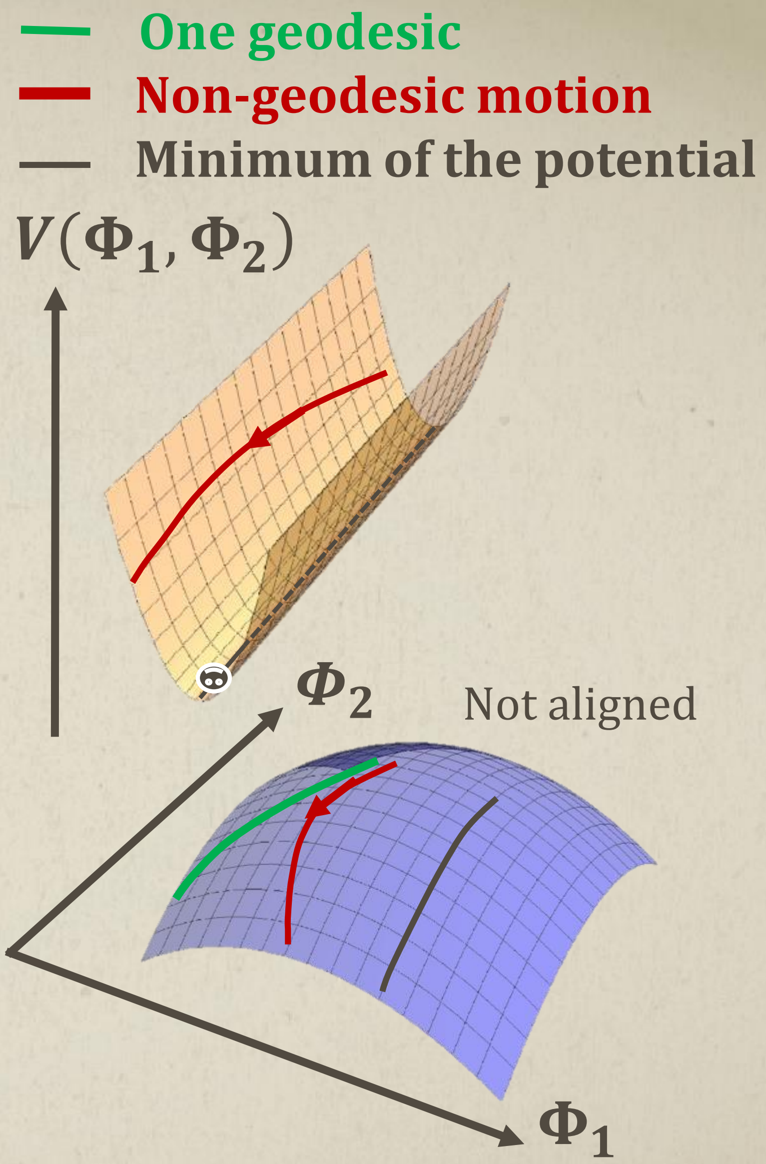
Why multifield?

- Single-field has theoretical limitations: η-problem, swampland conjectures, etc.
- Multifield inflation is expected from generic high-energy arguments, is more general, has a phenomenology that can be probed



Curved field space

Scalar curvature: $R_{fs} \neq 0$

STOCHASTIC FORMALISM FOR NON-LINEAR SIGMA MODELS

Massless + slow-variation

$$\left(\frac{H}{2\pi}\right)^2 G^{IJ}$$

- Slow-roll: $\frac{\mathcal{D}\phi^I_{IR}}{\mathrm{d}N} = -\left(\frac{G^{IJ}V_{,J}}{3H^2}\right)_{\phi_{IR}} + \Xi^I$ $\langle \Xi^I(N)\Xi^J(N')\rangle = P^{IJ}_{\phi_{UV}}\big(N, k_\sigma(N)\big)\delta(N-N')$

Covariant generalisation of single-field case

[L. Pinol, Y. Tada, S. Renaux-Petel 2018]
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Covariant generalisation of single-field case

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- New difficulties:

vielbeins: not unique

  - Find a square-root of the noise amplitude matrix: $G^{IJ} = g^I_\alpha g^J_\alpha$
  - Enforce covariance of the equations (also in SF, but more visible in MF): Itô calculus & the standard chain rule
  - Links with the discretisation ambiguity: choice of α more critical

FOKKER-PLANCK EQUATION FOR NON-LINEAR SIGMA MODELS

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$$\frac{\mathcal{D}\phi^I_{IR}}{\mathrm{d}N} = -\left(\frac{G^{IJ}V_{,J}}{3H^2}\right)_{\phi_{IR}} + \left(\frac{H}{2\pi}\,\boldsymbol{g^I_\alpha}\right)_{\phi_{IR}} \xi^\alpha$$

↓

Massless + slow-variation

$$\langle \xi^\alpha(N)\xi^\beta(N')\rangle = \delta^{\alpha\beta}\delta(N-N')$$

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$$\langle \xi^\alpha(N)\xi^\beta(N')\rangle = \delta^{\alpha\beta}\delta(N-N')$$

$$\frac{\partial \boldsymbol{P}}{\partial N} = D_I\left(\frac{G^{IJ}V_{,J}}{3H^2}\boldsymbol{P}\right) + \alpha D_I\left[\frac{H}{2\pi}g^I_\alpha D_J\left(\frac{H}{2\pi}g^J_\alpha \boldsymbol{P}\right)\right] + \left(\alpha-\frac{1}{2}\right)\frac{1}{\sqrt{G}}\partial_I\partial_J\left[\sqrt{G}\left(\frac{H}{2\pi}\right)^2 G^{IJ}\boldsymbol{P}\right]$$

Classical drift — Noise-induced drift + diffusion — Extra diffusion

NB: Here $P_s = P/\sqrt{G}$

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g^I_α: not unique

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Massless + slow-variation

$$\langle \xi^\alpha(N)\xi^\beta(N')\rangle = \delta^{\alpha\beta}\delta(N-N')$$

↓

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Classical drift — Noise-induced drift + diffusion — Extra diffusion: **not covariant!**

$\boldsymbol{D_I}X^J = \boldsymbol{\partial_I}X - \boldsymbol{\Gamma^J_{IK}}X^K$: covariant field-space derivative

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Choice of α crucial

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Choice of $\boldsymbol{\alpha}$ crucial

ITO AND STRATONOVICH

Itô, $\alpha = 0$

$$\partial_N P = -D_I(h^I P) + \frac{1}{2\sqrt{G}}\partial_I \partial_J \left(\sqrt{G} \left(\frac{H}{2\pi}\right)^2 g^I_\alpha g^J_\alpha P \right)$$

Stratonovich, $\alpha = 1/2$

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ITO AND STRATONOVICH

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- **Not covariant** under field redefinitions
- **No dependence** on the choice of vieilbeins:

$g^I_\alpha \rightarrow \Omega^\beta_\alpha \tilde{g}^I_\beta$ with $\Omega^\gamma_\alpha \Omega^\beta_\gamma = \delta^{\alpha\beta}$ does not change the physics

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- **Well covariant** under field redefinitions
- **Dependence** on the arbitrary choice of vieilbeins through derivatives of the orthogonal matrix $\boldsymbol{\Omega}$

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- **Well covariant** under field redefinitions
- **Dependence** on the arbitrary choice of vieilbeins

through derivatives of the orthogonal matrix $\boldsymbol{\Omega}$

No choice seems good

No hint from the derivation

Well-known problem in statistical physics of non-covariance of Itô calculus

- Fact about covariance: although X^I is well a covariant object, $\mathrm{d}X^I$ is not. *like u^μ vs $\mathrm{d}u^\mu$ in GR*

→ Define $\boldsymbol{\mathcal{D}}X^I = \mathrm{d}X^I + \Gamma^I_{JK}\partial_N\phi^J_{IR}X^K$

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→ Define $\mathcal{D}X^I = \mathrm{d}X^I + \Gamma^I_{JK}\partial_N\phi^J_{IR}X^K$ **Covariant in ODE and Stratonovich scheme, but not in Itô!**

- This is because the standard chain rule gets modified by noise^2 $\propto \Delta N$ terms (which **precisely cancel** only in Stratonovich $\boldsymbol{\alpha = 1/2}$)

$$\mathrm{d}f(N, X^I) = \left[\frac{\partial f}{\partial N} + \left(\frac{1}{2} - \boldsymbol{\alpha}\right)\frac{\partial^2 f}{\partial X^I \partial X^J} A^{IJ}_{XX}\right]\mathrm{d}N + \frac{\partial f}{\partial X^I}\mathrm{d}X^I$$

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- You should define a field-space covariant derivative in the sense of Itô as: *amplitude of* $\langle \Xi^J \Xi^K \rangle$

Itô calculus: **[Graham 1974]** $\mathfrak{D}X^I = \mathcal{D}X^I + \frac{1}{2}\Gamma^I_{JK}A^{JK}_{XX}\mathrm{d}N$

But we do not get such derivatives from the derivation of the Langevin equations…

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But we do not get such derivatives from the derivation of the Langevin equations…

Only Stratonovich choice respects covariance of the equations!

How to solve the ambiguity of the vielbein's choice?

QUANTISATION OF THE MULTIFIELD SYSTEM

Spoiler: there is a preferred frame for the noise diagonalisation → fundamental quantum oscillators

$$\hat{\phi}^I_{\vec{k}} = \phi^I_{k,\alpha}(N)\hat{a}^{\alpha}_{\vec{k}} + \left(\phi^I_{k,\alpha}\right)^*(N)\hat{a}^{\alpha,\dagger}_{-\vec{k}}$$

[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
Journal of Cosmology and Astroparticle Physics, JCAP04(2021)048

- Each quantum field is decomposed into the basis $\left(\hat{a}^{\alpha}_{\vec{k}}, \hat{a}^{\alpha,\dagger}_{-\vec{k}}\right)$ with mode decomposition $\phi^I_{k,\alpha}(N)$
- This basis is unique up to an irrelevant unitary matrix

QUANTISATION OF THE MULTIFIELD SYSTEM

Spoiler: there is a preferred frame for the noise diagonalisation → fundamental quantum oscillators

$$\hat{\phi}^I_{\vec{k}} = \phi^I_{k,\alpha}(N)\hat{a}^\alpha_{\vec{k}} + \left(\phi^I_{k,\alpha}\right)^*(N)\hat{a}^{\alpha,\dagger}_{-\vec{k}}$$

[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
Journal of Cosmology and Astroparticle Physics, JCAP04(2021)048

- Each quantum field is decomposed into the basis $\left(\hat{a}^\alpha_{\vec{k}}, \hat{a}^{\alpha,\dagger}_{-\vec{k}}\right)$ with mode decomposition $\phi^I_{k,\alpha}(N)$
- This basis is unique up to an irrelevant unitary matrix
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- The power spectra are actually $P^{IJ}_{\phi_{UV}}\left(N, k_\sigma(N)\right) = \sum_\alpha \phi^I_{k_\sigma,\alpha}(N)\left(\phi^J_{k_\sigma,\alpha}\right)^*(N)$
- Remember classicalisation: $\left(\phi^J_{k_\sigma(N),\alpha}\right)^*(N) \simeq \phi^J_{k_\sigma(N),\alpha}$ so the mode functions are the vielbeins:

$$\Xi^I(N) = \phi^I_{k_\sigma,\alpha}(N)\xi^\alpha, \qquad \text{with} \quad \left\langle \xi^\alpha(N)\xi^\beta(N')\right\rangle = \delta^{\alpha\beta}\delta(N-N')$$

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Moral

In practice you do not have the choice!

The square-root matrix must be the mode functions themselves:

$$\frac{H}{2\pi} g^I_\alpha = \phi^I_{k_\sigma,\alpha}(N)$$

No more ambiguity in the Stratonovich picture

RESOLUTION OF THE ANOMALIES

The circle denotes the Stratonovich prescription

$$\frac{\mathcal{D}\phi_{IR}^I}{\mathrm{d}N} = -\left(\frac{G^{IJ}V_{,J}}{3H^2}\right)_{\phi_{IR}} + \left(\phi^I_{k_\sigma,\alpha}\right)_{\phi_{IR}} \circ \xi^\alpha, \qquad \langle \xi^\alpha(N)\xi^\beta(N')\rangle = \delta^{\alpha\beta}\delta(N-N')$$

Seen as a function of $\boldsymbol{\phi}_{IR}(N)$ at the same time, otherwise non-Markovian

You can write the corresponding equation in the Itô prescription, if you add consistently the noise-induced drift

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Seen as a function of $\boldsymbol{\phi}_{IR}(N)$ at the same time, otherwise non-Markovian

You can write the corresponding equation in the Itô prescription, if you add consistently the noise-induced drift

→ Exactly what is needed to define the Graham, **Ito-covariant derivatives**!

$$\frac{\mathfrak{D}\phi^I_{IR}}{\mathrm{d}N} = -\left(\frac{G^{IJ}V_{,J}}{3H^2}\right)_{\phi_{IR}} + \frac{H[\boldsymbol{\phi}_{IR}(N)]}{2\pi} g^I_\alpha \cdot \xi^\alpha,$$

The dot denotes the Itô prescription

[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
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In practice, it is necessary to use the Itô prescription:
Now that it is covariant, you can forget the mode functions

FURTHER: PHASE-SPACE DYNAMICS

[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
Journal of Cosmology and Astroparticle Physics, JCAP04(2021)048

Beyond slow-roll, beyond massless ($a^3\pi$ conjugate momentum of ϕ)

$$\begin{cases} \dfrac{\mathfrak{D}\phi^I_{IR}}{\mathrm{d}N} = \dfrac{G^{IJ}\pi^{IR}_J}{H} + \Xi^I_\phi \\ \dfrac{\mathfrak{D}\pi^{IR}_I}{\mathrm{d}N} = -3H\pi^{IR}_I - \dfrac{V_{,I}}{H} + \Xi^\pi_I \end{cases}$$

$$\langle \Xi^I_\phi(N)\Xi^J_\phi(N')\rangle = \left(P_{\phi_{UV},\phi_{UV}}\right)^{IJ}(N, k_\sigma(N))\delta(N-N')$$

$$\langle \Xi^I_\phi(N)\Xi^\pi_J(N')\rangle = \left(P^{\pi_{UV}}_{\phi_{UV}}\right)^I_J(N, k_\sigma(N))\delta(N-N')$$

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+ analytical formulas for the noise amplitude in a slow-varying approximation

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amplitude of $\langle \Xi^J_X \Xi^K_X\rangle$

Itô calculus in field space: **[Graham 1974]** $\mathfrak{D}X^I = \mathcal{D}X^I + \frac{1}{2}\Gamma^I_{JK}A^{JK}_{XX}\mathrm{d}N$

Itô calculus in phase space: **[L. Pinol, S. Renaux-Petel, Y. Tada 2020]** $\mathfrak{D}V_I = \mathcal{D}V_I - \frac{1}{2}\left(\Gamma^S_{IJ,K} + \Gamma^S_{KM}\Gamma^M_{IJ}\right)V_S A^{JK}_{XX}\mathrm{d}N - \Gamma^K_{IJ}\left(A^V_X\right)^J_{\ K}\mathrm{d}N$

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+ analytical formulas for the noise amplitude in a slow-varying approximation

Remark: cross-power spectra $\left(P_{\phi_{UV}}^{\pi_{UV}}\right)^I_J$ have a non-zero imaginary part fixed by the commutation relations

$$\left[\phi^I_{UV}(x), a^3\pi_J^{UV}(y)\right] = i\hbar\delta^{(4)}(x-y) \rightarrow \mathrm{Im}\left[\left(P_{\phi_{UV}}^{\pi_{UV}}\right)^I_J\right] \propto 1/a^3$$ but non-zero…

→ Imaginary noise?

Conceptual issue resolved with path-integral approach

FURTHER: PHASE-SPACE DYNAMICS

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- Markovian approximation → Fokker-Planck equation
- Need to define field-derivatives (not time) that are phase-space covariant: $\mathcal{D}_{\phi^I_{IR}} U^J = D_{\phi^I_{IR}} U^J + \Gamma^K_{IL}\pi^{IR}_K \partial_{\pi^{IR}_L} U^I$

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We find a covariant multi-dimensional, phase-space Fokker-Planck equation beyond slow-roll massless fields:

$$\frac{\partial P}{\partial N} = -\mathcal{D}_{\phi^I_{IR}}\left(\frac{G^{IJ}\pi_J^{IR}}{H}P\right) + \partial_{\pi_I^{IR}}\left[\left(3\pi_I^{IR} + \frac{V_I}{H}\right)P\right] + \frac{1}{2}\mathcal{D}_{\phi^I_{IR}}\mathcal{D}_{\phi^J_{IR}}\left[\left(P_{\phi_{UV},\phi_{UV}}\right)^{IJ}P\right]$$

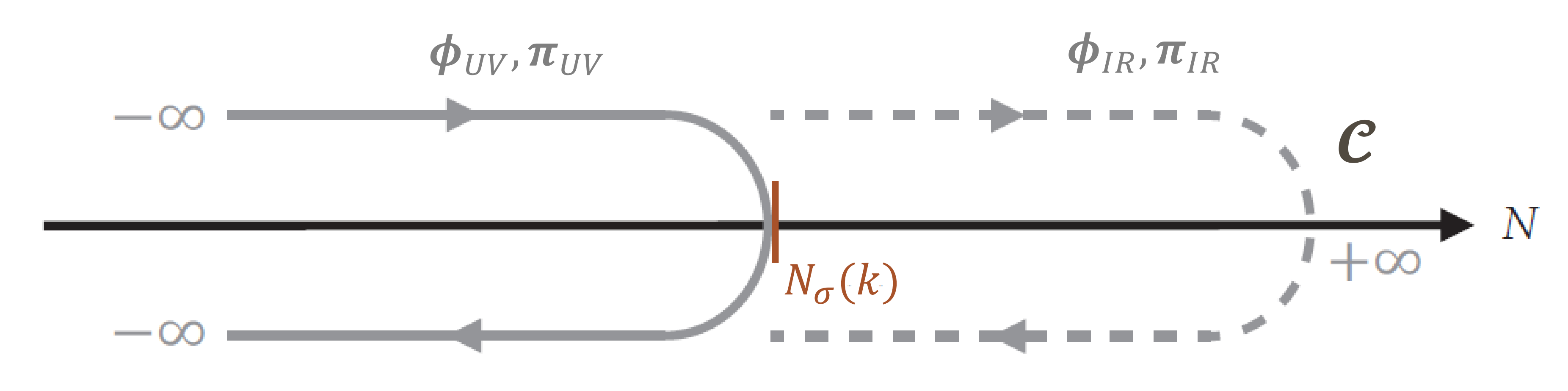
$$+\mathcal{D}_{\phi^I_{IR}}\partial_{\pi_J^{IR}}\left[\left(P^{\pi_{UV}}_{\phi_{UV}}\right)^I_{\ J}P\right] + \frac{1}{2}\partial_{\pi_I^{IR}}\partial_{\pi_J^{IR}}\left[\left(P^{\pi_{UV},\pi_{UV}}\right)_{IJ}P\right]$$

$P(\boldsymbol{\phi}_{IR},\boldsymbol{\pi}^{IR},N)$

IV. PATH-INTEGRAL DERIVATION

[L. Pinol, S. Renaux-Petel, Y. Tada 2020]
Journal of Cosmology and Astroparticle Physics, JCAP04(2021)048

The Schwinger-Keldysh formalism…
…for cosmology



If time permits only…

A CLOSED TIME PATH

- We want to find the stochastic formalism for multifield inflation in **phase space** from a path-integral approach

Generating functional: $$Z[J_{XI}] = \int_{\mathcal{C}} \mathcal{D}\phi^{XI} \exp\left[iS[\phi^{XI}] + i\int \mathrm{d}^4x\, J_{XI}\phi^{XI}\right]$$

Multifield **Hamiltonian** action: $\phi^{XI} = (\phi^I, \pi_I)$

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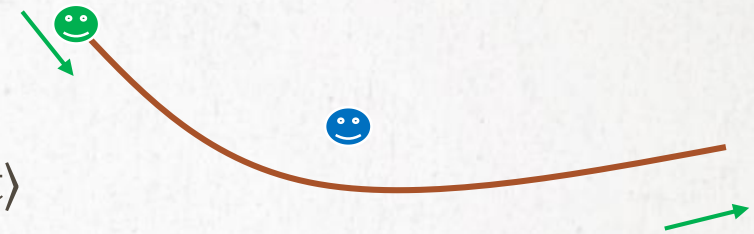

  - Particle physics: in-out, S-matrix scattering <u>amplitudes</u> $\langle \mathrm{in}|\hat{S}|\mathrm{out}\rangle$

asymptotic non-interacting states: in=past ; out=future

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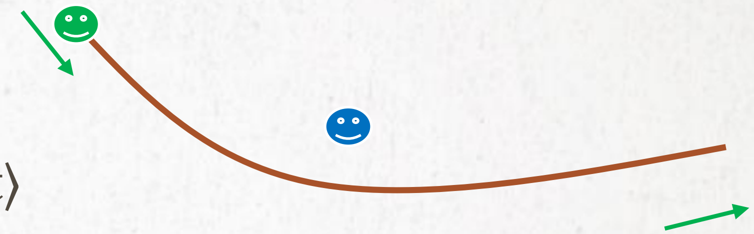

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    ↳ We compute in-in, time-dependent <u>correlators</u>: $\langle \mathrm{in}|\hat{O}(t_\star)|\mathrm{in}\rangle$

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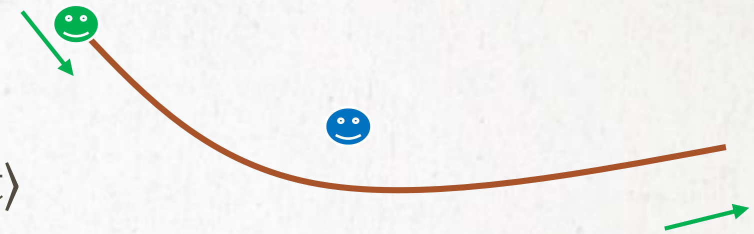

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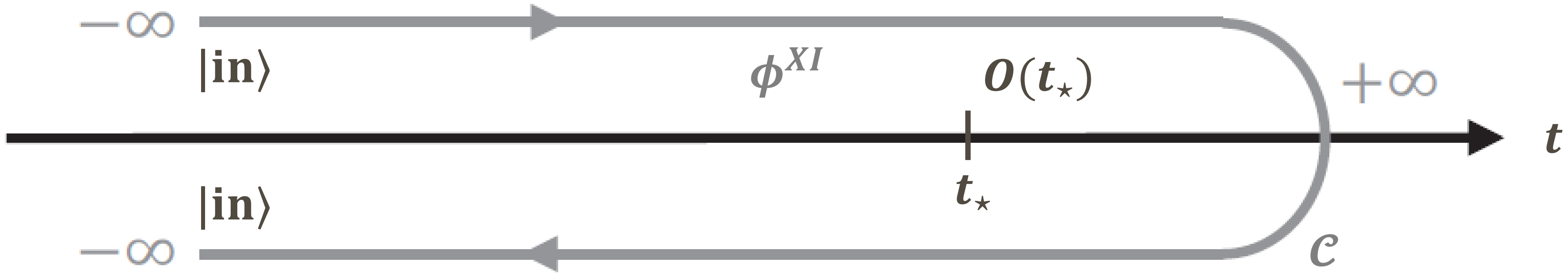
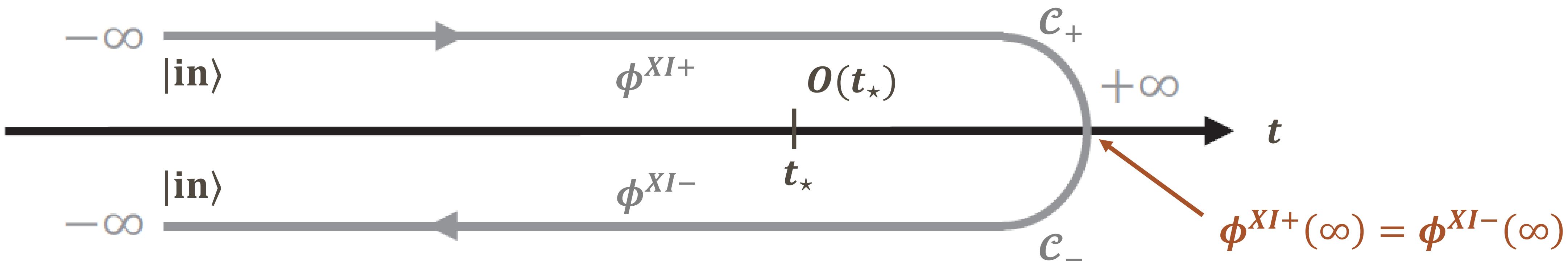


SCHWINGER-KELDYSH FORMALISM

- We split the contour of integration: $\mathcal{C} = \mathcal{C}_+ + \mathcal{C}_-$ and denote the fields on each branch ϕ^{XI+} and ϕ^{XI-}

Generating functional: $$Z[J_{XI\pm}] = \int_{\mathcal{C}_+} \mathcal{D}\phi^{XI\pm} \exp\left[i(S[\phi^{XI+}] - S[\phi^{XI-}]) + i\int \mathrm{d}^4x\,(J_{XI+}\phi^{XI+} - J_{XI-}\phi^{XI-})\right]$$

Schwinger-Keldysh formalism: doubling of the d.o.f.



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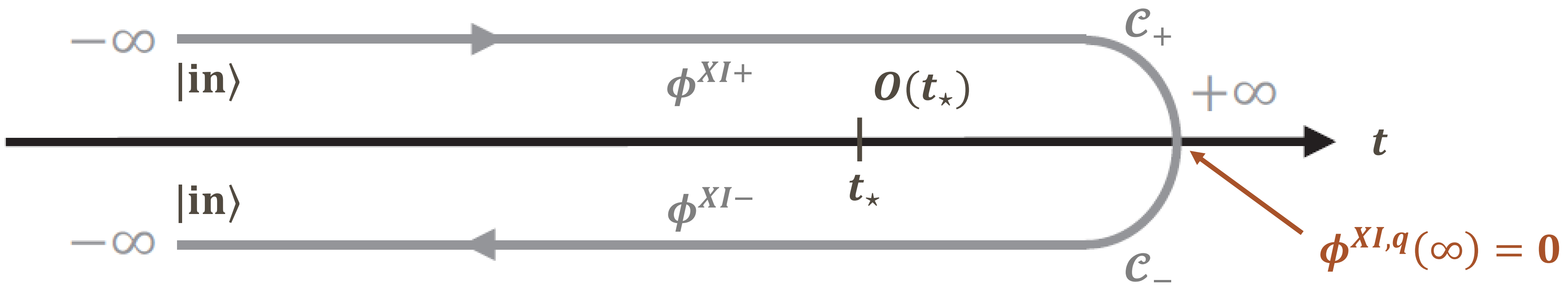
- Keldysh basis: $\phi^{XI,\mathrm{cl}} = \dfrac{\phi^{XI+} + \phi^{XI-}}{2}$; $\phi^{XI,\mathrm{q}} = \phi^{XI+} - \phi^{XI-}$

Same history: classical *Difference of histories: stochasticity / quantumness*



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Same history: classical — *Difference of histories: stochasticity / quantumness*

- Theory in the Keldysh basis $$Z[J_{XIa}] = \int_{\mathcal{C}_+} \mathcal{D}\phi^{XIa} \exp\left[iS[\phi^{XIa}] + i\int \mathrm{d}^4x\, J_{XIa}\phi^{XIa}\right]$$

With $S[\phi^{XIa}] = S\left[\phi^{XI,\mathrm{cl}} + \frac{\phi^{XI,\mathrm{q}}}{2}\right] - S\left[\phi^{XI,\mathrm{cl}} - \frac{\phi^{XI,\mathrm{q}}}{2}\right]$ + rule of contraction $X_aY^a = \sigma^1_{ab}X^bY^a$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

AN EFFECTIVE, COARSE-GRAINED THEORY

- We split the fields in IR + UV: $\phi^{XIa} = \phi_{IR}^{XIa} + \phi_{UV}^{XIa}$ and integrate over the UV, small scales:

$$Z = \int \mathcal{D}\phi_{IR}^{XIa} \exp\left[iS_{\text{eff}}[\phi_{IR}^{XIa}]\right] , \text{ with } \exp(iS_{\text{eff}}[\phi_{IR}^{XIa}]) = \int \mathcal{D}\phi_{UV}^{XIa} \exp\left[iS[\phi_{IR}^{XIa} + \phi_{UV}^{XIa}]\right]$$

Coarse-grained effective action

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Coarse-grained effective action

- Can only be done perturbatively in ϕ_{UV}^{XIa}, but remember $\phi_{UV}^{XIa} \ll \phi_{IR}^{XIa}$: develop $S[\phi_{IR}^{XIa} + \phi_{UV}^{XIa}]$ at 2nd order

Use covariant Vilkoswky-de Witt variables

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Use covariant Vilkoswky-de Witt variables

$$S = S^{(0)} + S^{(1)} + S^{(2)} \quad \rightarrow \quad \text{Gaussian integral over } \phi_{UV}^{XIa}$$

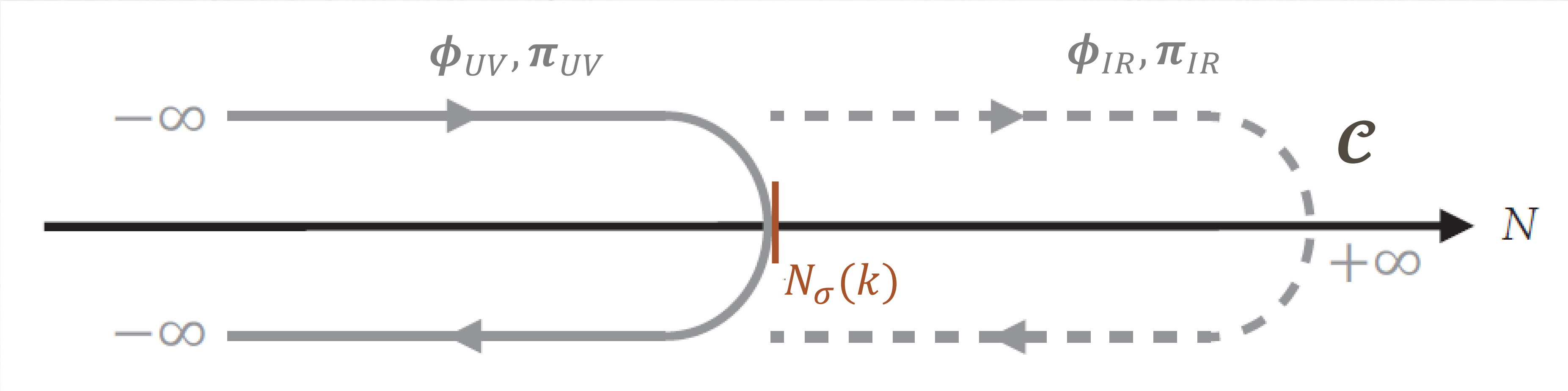
Background action

Linear coupling due to time-dependent cut-off

Quadratic action for perturbations in a background of IR fields

Crucial because otherwise the two sectors do not couple

LEADING-ORDER QUANTUM EFFECT



- Skipping many (thrilling) details, you get: $S_{\text{eff}}[\phi_{IR}^{XIa}] = \mathrm{S}^{(0)}[\phi_{IR}^{XIa}] + i\hbar \int d^4x\, a^3 \xi_{XI} \phi_{IR}^{XI,\mathrm{q}} + O\left[\left(\hbar\phi_{IR}^{XI,\mathrm{q}}\right)^2\right]$

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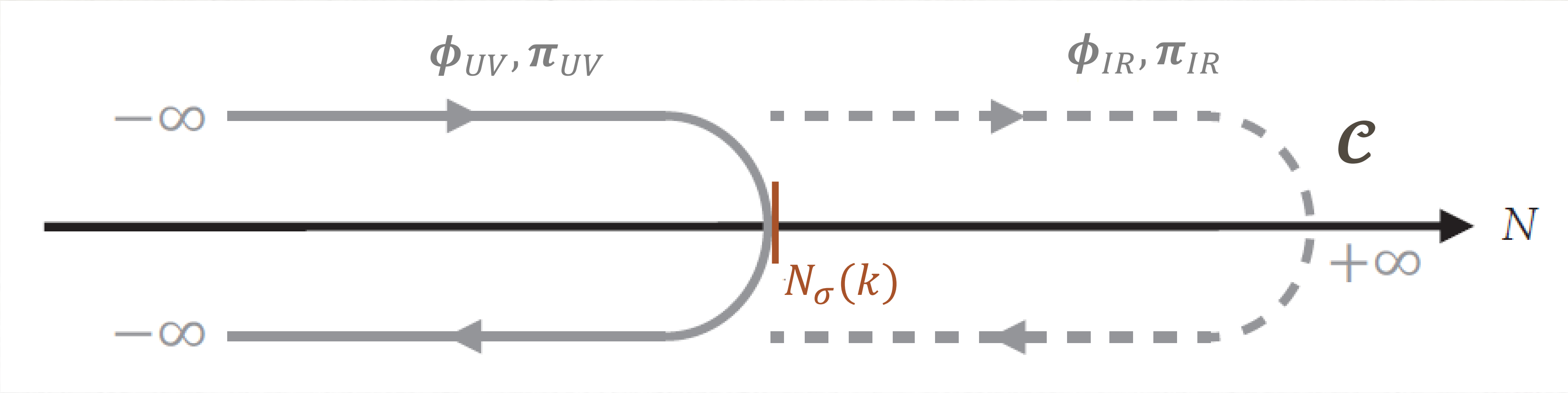

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with ξ_{XI} auxiliary Gaussian fields that verify: $\left\langle \xi_{XI}(x)\xi_{YJ}(x')\right\rangle = \textbf{Re}\left(\langle \phi_{UV}^{XI}(x)\phi_{UV}^{YJ}(x')\rangle\right)$

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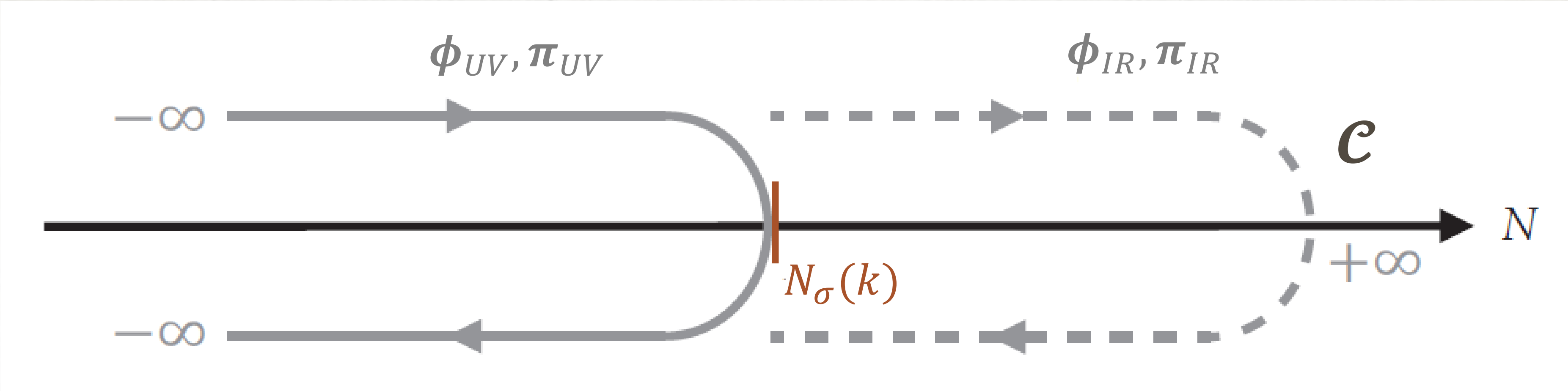

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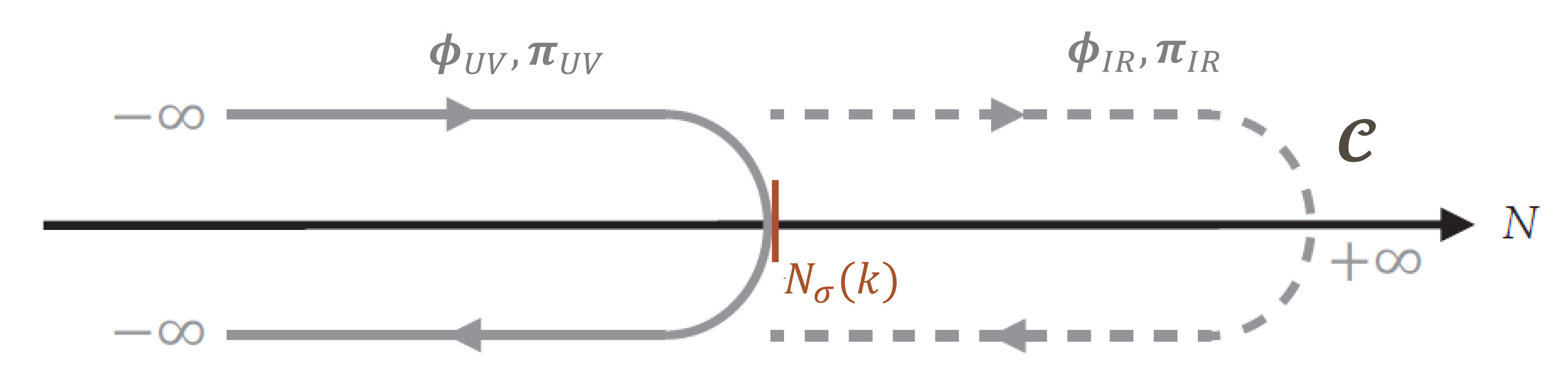
with ξ_{XI} auxiliary Gaussian fields that verify: $\langle \xi_{XI}(x)\xi_{YJ}(x')\rangle = \mathbf{Re}\left(\langle \phi_{UV}^{XI}(x)\phi_{UV}^{YJ}(x')\rangle\right)$

- Partition function: $Z = \int \mathcal{D}\phi_{IR}^{XIa} \exp\left[i \int \phi_{IR}^{XI,q} \left(\frac{\delta S^{(0)}[\phi_{IR}^{XIa}]}{\delta \phi_{IR}^{XI,q}}\bigg|_{\phi_{IR}^{XI,cl}} + a^3 \xi_{XI} \right) + O\left[\left(\hbar\phi_{IR}^{XI,q}\right)^2\right] \right]$

$\int \mathcal{D}\phi\, e^{i\int \phi K} = \delta(K) \rightarrow K$

LEADING-ORDER QUANTUM EFFECT



- Skipping many (thrilling) details, you get: $S_{\text{eff}}[\phi_{IR}^{XIa}] = S^{(0)}[\phi_{IR}^{XIa}] + i\hbar \int d^4x\, a^3 \xi_{XI} \phi_{IR}^{XI,\text{q}} + O\left[\left(\hbar\phi_{IR}^{XI,\text{q}}\right)^2\right]$

with ξ_{XI} auxiliary Gaussian fields that verify: $\langle \xi_{XI}(x)\xi_{YJ}(x')\rangle = \textbf{Re}\left(\langle \phi_{UV}^{XI}(x)\phi_{UV}^{YJ}(x')\rangle\right)$

- Partition function: $Z = \int \mathcal{D}\phi_{IR}^{XIa} \exp\left[i \int \phi_{IR}^{XI,\text{q}} \left(\frac{\delta S^{(0)}[\phi_{IR}^{XIa}]}{\delta \phi_{IR}^{XI,\text{q}}}\Big|_{\phi_{IR}^{XI,\text{cl}}} + a^3 \xi_{XI} \right) + O\left[\left(\hbar\phi_{IR}^{XI,\text{q}}\right)^2\right] \right]$

$\int \mathcal{D}\phi\, e^{i\int \phi K} = \delta(K) \rightarrow K$

↳ only contributing trajectories are the « classical ones »: $\frac{\delta S^{(0)}[\phi_{IR}^{XIa}]}{\delta \phi_{IR}^{XI,\text{q}}}\Big|_{\phi_{IR}^{XI,\text{cl}}} + a^3 \xi_{XI} = 0$

THE LANGEVIN EQUATIONS

The classical EoM of the effective action includes the first quantum corrections:

$$\left.\frac{\delta S^{(0)}[\phi_{IR}^{XIa}]}{\delta \phi_{IR}^{XI,\mathrm{q}}}\right|_{\phi_{IR}^{XI,\mathrm{cl}}} + \boldsymbol{a^3 \xi_{XI}} = 0$$

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$$\begin{cases} \dfrac{\mathcal{D}\phi_{IR}^{I}}{\mathrm{d}N} = \dfrac{G^{IJ}\pi_J^{IR}}{H} + \Xi_\phi^I \\ \dfrac{\mathcal{D}\pi_I^{IR}}{\mathrm{d}N} = -3H\pi_I^{IR} - \dfrac{V_{,I}}{H} + \Xi_I^\pi \end{cases}$$

Stratonovich

$$\langle \Xi_\phi^I(N)\Xi_\phi^J(N')\rangle = \left(P_{\phi_{UV},\phi_{UV}}\right)^{IJ}(N,k_\sigma(N))\delta(N-N')$$

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Itô

CONCLUSION

- Stochastic formalism for inflation: an effective theory for largest cosmological scales
- Enables to derive non-perturbative results, to resum IR divergencies of QFT in de Sitter, etc.
- We extended it to multifield models and to phase-space dynamics beyond slow-roll massless fields
- We unveiled inflationary stochastic anomalies related to the discretisation ambiguity of SDEs
- We showed how to solve the anomalies by choosing the Stratonovich scheme and the frame of fundamental quantum oscillators to diagonalise the noise amplitude
- We used the Schwinger-Keldysh formalism borrowed from thermal field theory

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$$\hookrightarrow \quad \left.\frac{\delta S^{(0)}[\phi_{IR}^{XIa}]}{\delta \phi_{IR}^{XI,\mathrm{q}}}\right|_{\phi_{IR}^{XI,\mathrm{cl}}} + a^3 \xi_{XI} + O\left(\phi_{IR}^{XI,q}\right) = 0$$

Next-order correction